Families without minimal numberings

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A mapping $\alpha : \omega \longrightarrow \mathcal{A}$ of the set ω of natural numbers onto a family $\mathcal{A}$ of c.e. sets is called a *computable numbering* of $\mathcal{A}$ if the set $\{\langle x, n\rangle \mid x \in \alpha(n)\}$ is c.e. And a family $\mathcal{A}$ of subsets of ω is called *computable* if it has a computable numbering.

A numbering α is called *reducible* to a numbering β (in symbols, $\alpha \leqslant \beta$) if $\alpha = \beta \circ f$ for some computable function f. Two numberings α, β are called $equivalent$ if they are reducible to each other.

by θ_α we denoted the enumeration equivalence $\{< x, y >| \ \alpha x = \alpha y\}$ of a enumeration α.we denote by W_n and φ_m respectively the r.e set with the Post number n and the unary p.r.f.with the Kleene number *m*.

By π we denote the principal computable enumeration of all computable enumeration of families of r.e.sets; $\pi_n(x)$ denotes the x-th r.e set in the enumeration π_n. The principal computable enumeration of the family of all positive equivalences is denoted by ε.

For a computable enumeration α we denote by α_x^t the finite pair of the set αx that has been enumerated by the end of stage t in a certain fixed uniform enumeration of the family $\{\alpha x | x \in N\}$ in the case when α is given enumeration, or the set of elements enumerated in αx by the end of stage t of the construction when α is the currently constructed enumeration. we use similar notations for r.e.sets and p.r.f.'s.

Let $A(m, x, t)$ denote a function satisfying the following conditions:

1. $\text{range}(A) \subseteq \{0, 1\}$;

2. $A(e, x, 0) = 0$, for all e and x.

we have the computation function that $A(m,x,t)$, there exist r.f. $\Pi(e,l,y,t)$ for every e,l,y and this function consider following condition:

1. $\Pi(e,l,y,0) = 0$

2. $|\{t|\Pi(e,l,y,t+1) \neq \Pi(e,l,y,t)\}| \leq i$ where $i \geq 1$ is the Ershov's Hierarchy degree.

Σ_n^{-1} – the class of level n of the *Ershov hierarchy* of sets. Σ_n^0 – the class of level n of the arithmetical hierarchy.

The notion of a computable numbering for a family $\mathcal{A}$ of sets in the class Σ_n^i. may be deduced from the Goncharov–Sorbi approach as follows:

A numbering α of a family $\mathcal{A} \subseteq \Sigma_n^i$ is Σ_n^i-computable if $\{\langle x,m\rangle : x \in \alpha(m)\} \in \Sigma_n^i$, i.e. the sequence $\alpha(0), \alpha(1), \ldots$ of the members of $\mathcal{A}$ is uniformly Σ_n^i.

Computable numberings of the families of sets from any given level of the arithmetical or analytical hierarchy as well as the hierarchy of Ershov are usually identified with the uniformly enumerable sequences of sets from that level. More precisely, if $\{\mathcal{H}_n\}_{n\in\omega}$ is one of these hierarchies then a sequence A_0, A_1, $A_2, \ldots$ of the subsets of ω from $\mathcal{H}_n$ is uniformly enumerable if the set $\{\langle x, k\rangle : x \in A_k\}$ is in $\mathcal{H}_n$.

The set of all computable numberings of $\mathcal{A}$ is reordered by the reducibility relation of numberings.Minimal numberings of $\mathcal{A}$ are exactly those which are minimal in this reorder. It is easy to see that any finite family of the sets from any mentioned above hierarchy has a numbering which is reducible to any numbering of the family.

Theorems on an existence of the families without computable minimal numberings are analogs of the well known speed up theorems. The families of c.e. sets without computable minimal numberings were built in [1],[2].

[1]:V.V. V'yugin. On some examples of upper sublattices of computable enumerations,Algebra and Logic.vol. 12 (1973), no. 5, pp. 287–296.

[2]:S.A. Badaev ,On minimal enumerations,Siberian Advances in Mathematics.vol. 2 (1992), no. 1, pp. 1–30.

Theorem For every finite level of the hierarchy of Ershov there exists a computable family of sets from this level which has no any computable minimal numbering.

Conventions: we use the priority method is most of our proofs. At every stage of the priority construction we consider several cases. we assume that we check these cases one by one . if the conditions of none of these cases are satisfied , then we automatically pass to the next stage .

otherwise we execute the instructions of the first case for which the conditions are satisfied and pass to the next stage immediately , if the case are divided into some subcase , then we act in similar way.

if at stage $t+1$ either some set A^{t+1} or the value of the currently constructed function is not defined until the end of the stage , then we assume by definition that $A^{t+1} = A^t, g(..., t+1) = g(..., t)$

If a number $a \in N$ is not placed into some set $\alpha x, x \in N$ which is currently constructed by a priority construction , then we call it *unused at stage* $t+1$

Proof We shall construct a certain enumeration α step by step. The family $\mathcal{A} \rightleftharpoons \{\langle x, n\rangle \mid x \in \alpha(n)\}$ will be the required one. while performing the construction, we aim at define two pair of auxiliary π_n-numbers $a(n,k)$, $b(n,k)$,$c(n,k)$,$d(n,k)$ for each pair of numbers $k, n \in N$.

Stage 0.
1). $A(m,x,0) = 0$, for every m,x; $c(n,k)$,$d(n,k)$ $x(n,k,t)$ and
$y(n,k,t) \uparrow$ are not defined . *Stage* t+1. we must look following several case,
let $n = l(l(t)), k = r(l(r))$, and $a(n,k) = 2 < n,k >, b(n,k) = 2 < n,k > +1$ for every n,k;
case 1.
$\alpha_a^t = \alpha_b^t = \emptyset$
in this case so $x(n,k,t) \uparrow$ and $y(n,k,t) \uparrow$ are not defined too

Put $\alpha^t{+}1_a = 2c(n,k), z_0, z_1, \alpha^t{+}1_b = 2c(n,k), z_2, z_3$, where z_0, z_1, z_2, z_3 (let $z_0 < z_1 < z_2 < z_3$)are the four minimal natural numbers unused at this stage, then $x(,n,k,t+1) \rightleftarrows z_0, y(n,k,t+1) \rightleftarrows z_1, u(n,k,t+1) \rightleftarrows z_2, v(n,k,t+1) \rightleftarrows z_3$, we did't change other function's means so,

$$(A(m,x,t+1) = A(m,x,t))$$

then,$A(a(n,k), z_j, t{+}1) = 1, j =, 0, 1, 2, 3$,other way,we have put the new element

case 2.

all the $x(n,k,t)$, $y(n,k,t)$,$u(n,k,t)$ and $v(n,k,t)$ are defined ,so $x(n,k,t) \downarrow= z_0$, $y(n,k,t) \downarrow= z_1$,$u(n,k,t) \downarrow= z_2$ $v(n,k,t) \downarrow= z_3$, but $c(n,k)$ and $d(n,k)$ are not defined yet,

so $\exists c, d$ such that

$\Pi(n, c, z_j, t) = 1$,

we defined $c(n,k) = c, d(n,k) = d$. we never change x and y unless c and d defined.

case 3.

$c(n,k)\downarrow$,$d(n,k)\downarrow$, so there $x = z_0, y = z_1, u = z_2, v = z_3$

subcase 3.1

$A(a(n,k), z_j, t) = 1, and \Pi(n, c(n,k), z_j, t) = 0$
where $j = \overline{0,3}$
or
$A(a(n,k), z_j, t) = 0, and \Pi(n, c(n,k), z_j, t) = 1$

subcase 3.2

$A(a(n,k),z_j,t)=1, and \Pi(n,c(n,k),z_j,t)=1$

and for the each z_0, z_1, z_2, z_3:

$\Pi(n,c(ord),z_0,s+1) \neq \Pi(n,c(ord),z_0,s)$;

and $s+1 \leq t$,

then we have

$A(a(n,k),z_j,t+1)=0$

Now we look that question two way , one of is that when i odd and when it is even

I: when i be even;

subcase 3.2.a:

$A(a(n,k),z_0,t)=1,\Pi(n,c(n,k),z_0,t)=1$

step 1.1: if level is less than i is true , then

$A(a(n,k),z_1,t)=1,\Pi(n,c(n,k),z_1,t)=1$

then

$A(a(n,k),z_1,t)=0,\Pi(n,c(n,k),z_1,t)=1$

after this we do operation 'kill' then $A(a(n,k),z_1,t)=0,\Pi(n,c(n,k),z_1,t)=0$

after that we come back to the procedure of check level;

step 1.2: $A(a(n,k),z_1,t) = 0, \Pi(n,c(n,k),z_1,t) = 1$
then this is the one of the case of step 1.1 , so we check next step.

step 1.3: if this is true, then $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t) = 0$
then we do operation "kill" . After than we will check the level. if the level is not equal to i, then we will do operation "live" then we have $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t) = 1$
after this we must check the level.

step 1.4: $A(a(n,k),z_1,t) = 0, \Pi(n,c(n,k),z_1,t) = 0$

we do kill, then $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t)$
0

, and then $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t) = 0$ after this we will do operation 'live',

so $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t) = 1$

after this we must check the level.

step2.1: if level is less than i is false , then:

case 2.1: if $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t) = 0$ we can do operation 'kill' and so;

case2.2: if $A(a(n,k),z_1,t) = 0, \Pi(n,c(n,k),z_1,t) = 0$ so we check that $< c,d > \in \varepsilon_k^t$.

If π_n is an enumeration of family A, then $< a(n,k).b(n,k) > \in \varepsilon_k$ if and only if $\alpha a \neq \alpha b$.

If $< a(n,k),b(n,k) > \in \varepsilon_k^{t_0}$ for some t_0, then case 3 does not occur at stage $t+1$ for all $t \geq t_0, t+1 \in T(n,k)$. Hence $\alpha a = \alpha b$

II:when i be odd and level is less than i is true , then

step 2.1 : $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t) =$ 1 then
$A(a(n,k),z_1,t) = 0, \Pi(n,c(n,k),z_1,t) = 1$;
we do operation kill: $A(a(n,k),z_1,t) = 0, \Pi(n,c(n,k),$
0 ; we must do operation 'live',so $A(a(n,k),z_1,t) =$ $1, \Pi(n,c(n,k),z_1,t) = 1$, after that come back to check level

step 2.2: $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,t) =$ 0 ; operation kill: $A(a(n,k),z_1,t) = 1, \Pi(n,c(n,k),z_1,$
1,then do operation 'live' ,after that we check level.

step 2.3: $A(a(n,k),z_1,t) = 0, \Pi(n,c(n,k),z_1,t) =$ 1, this is the subcase of case 2.1;

step 2.4: $A(a(n,k),z_1,t) = 0, and \Pi(n,c(n,k),z_1,t) =$ 0 this is the subcase of case 2.1;

step3.c: if level is less then i is false , then we must check other enumeration and come back to the check level
case 3.1: if the level is equal to the i then also check the all the level.

Thank you for your attention!

Enjoy Banquet!