Lexicographic products of modal logics with linear frames

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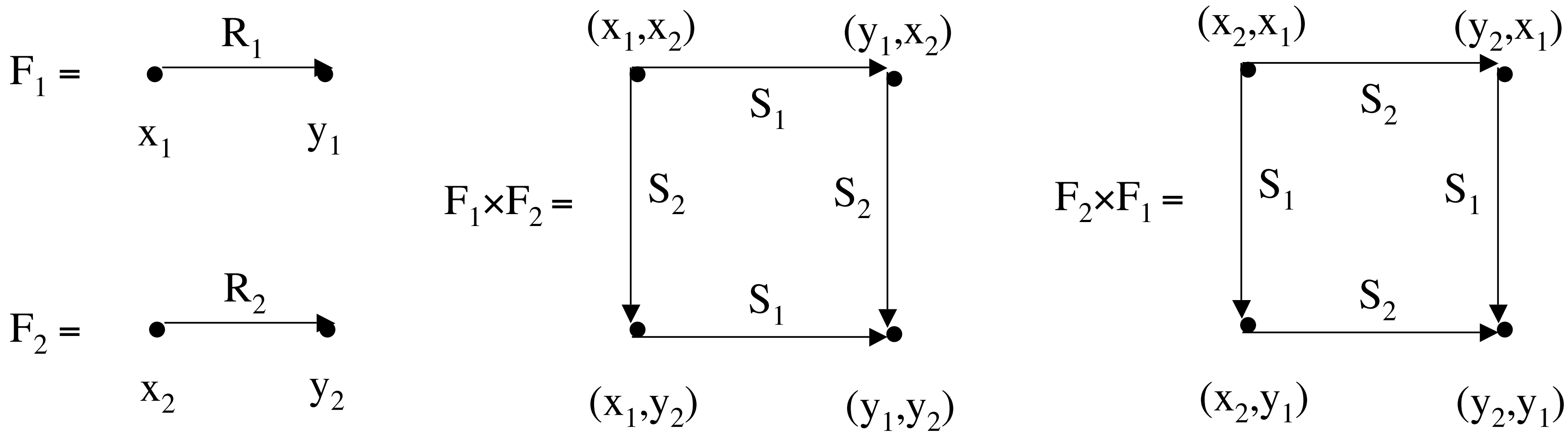
Products of relational structures

- Asynchronous products of relational structures
  - $F_1 = (W_1,R_1)$, $F_2 = (W_2,R_2)$
  - $F_1 \times F_2 = (W,S_1,S_2)$ where
    - $W = W_1 \times W_2$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1\ R_1\ y_1$ and $x_2 = y_2$
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_1 = y_1$ and $x_2\ R_2\ y_2$

Products of relational structures

- Asynchronous products of relational structures



Products of relational structures

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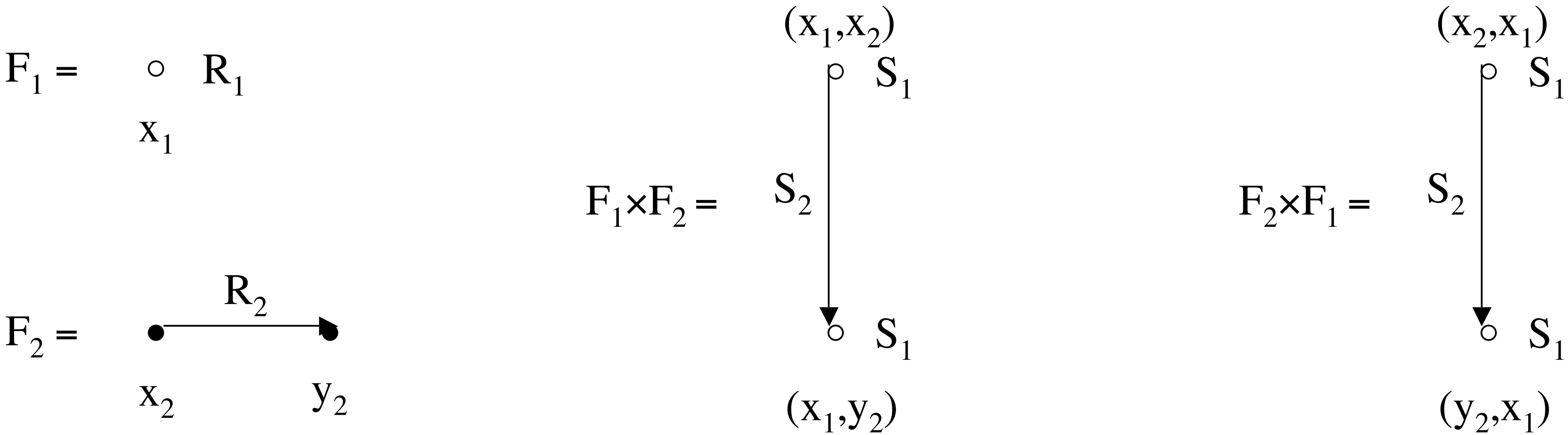

Products of relational structures

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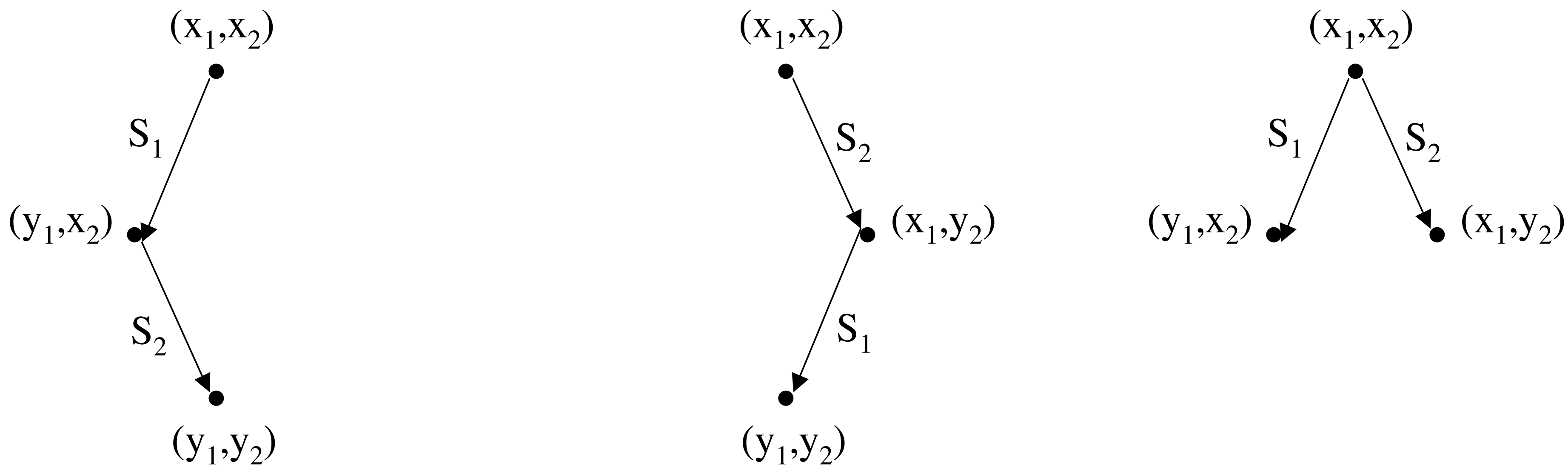
  - $F_1 = (W_1,R_1)$, $F_2 = (W_2,R_2)$, $F_1 \times F_2 = (W,S_1,S_2)$



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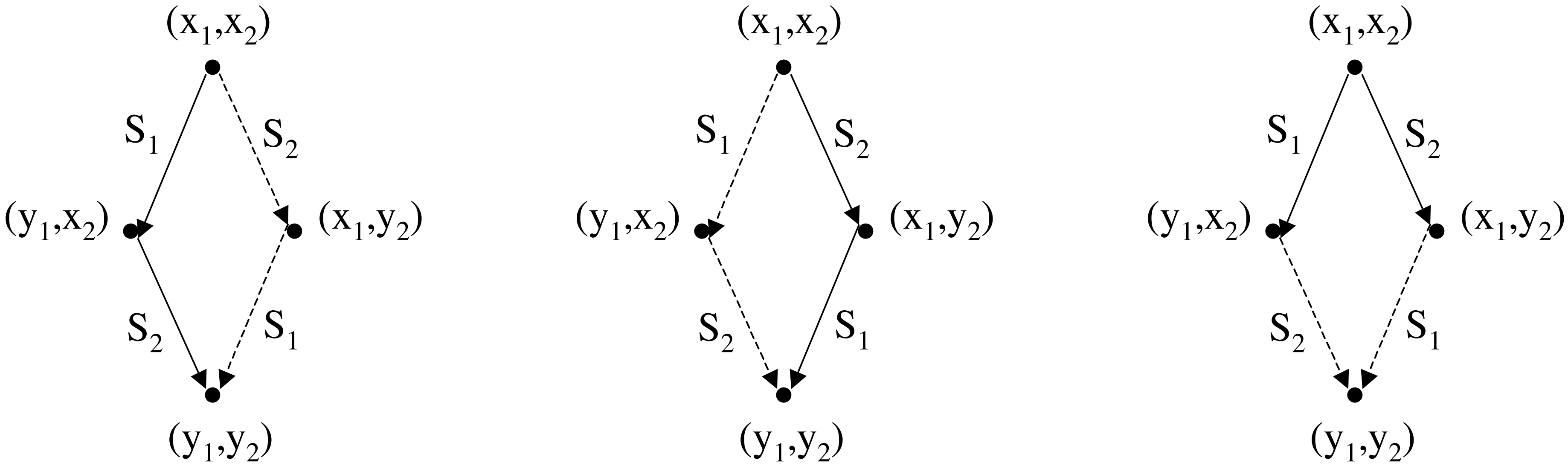


Products of relational structures

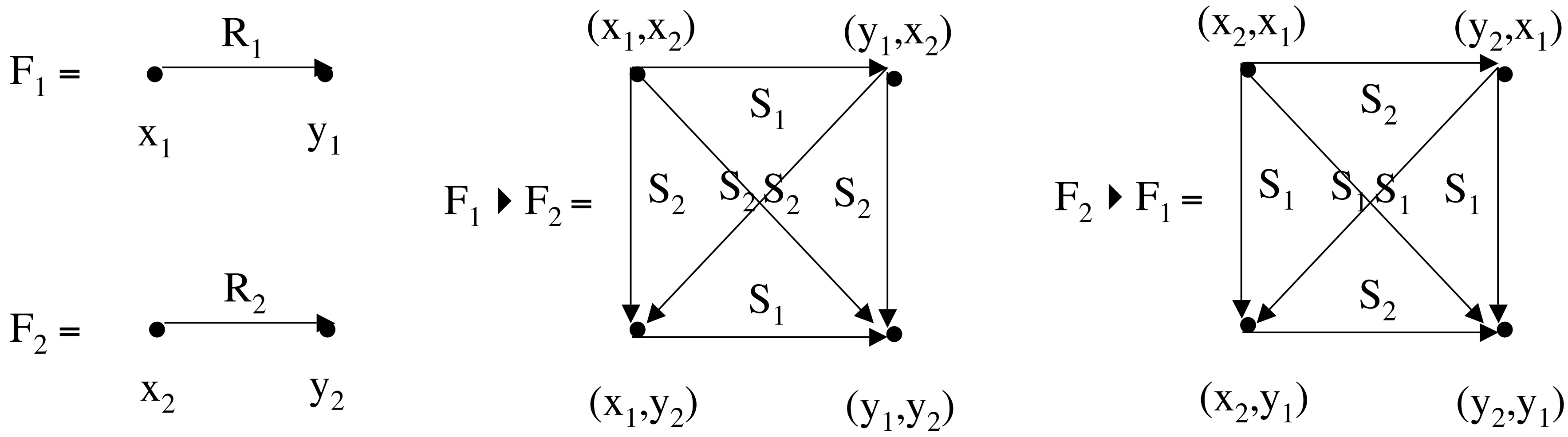
- Asynchronous products of relational structures
  - Let $F = (W,S_1,S_2)$ be countable and such that
    - $\forall x \forall y(\exists z(xS_1z \ \& \ zS_2y) \Rightarrow \exists z(xS_2z \ \& \ zS_1y))$
    - $\forall x \forall y(\exists z(xS_2z \ \& \ zS_1y) \Rightarrow \exists z(xS_1z \ \& \ zS_2y))$
    - $\forall x \forall y(\exists z(zS_1x \ \& \ zS_2y) \Rightarrow \exists z(xS_2z \ \& \ yS_1z))$
  - Then there exists $F_1 = (W_1,R_1)$ and $F_2 = (W_2,R_2)$ such that F is a p-morphic image of $F_1 \times F_2$

Products of relational structures

- Lexicographic products of relational structures
  - $F_1 = (W_1,R_1)$, $F_2 = (W_2,R_2)$
  - $F_1 \blacktriangleright F_2 = (W,S_1,S_2)$ where
    - $W = W_1 \times W_2$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1\ R_1\ y_1$ and $x_2 = y_2$
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_2\ R_2\ y_2$

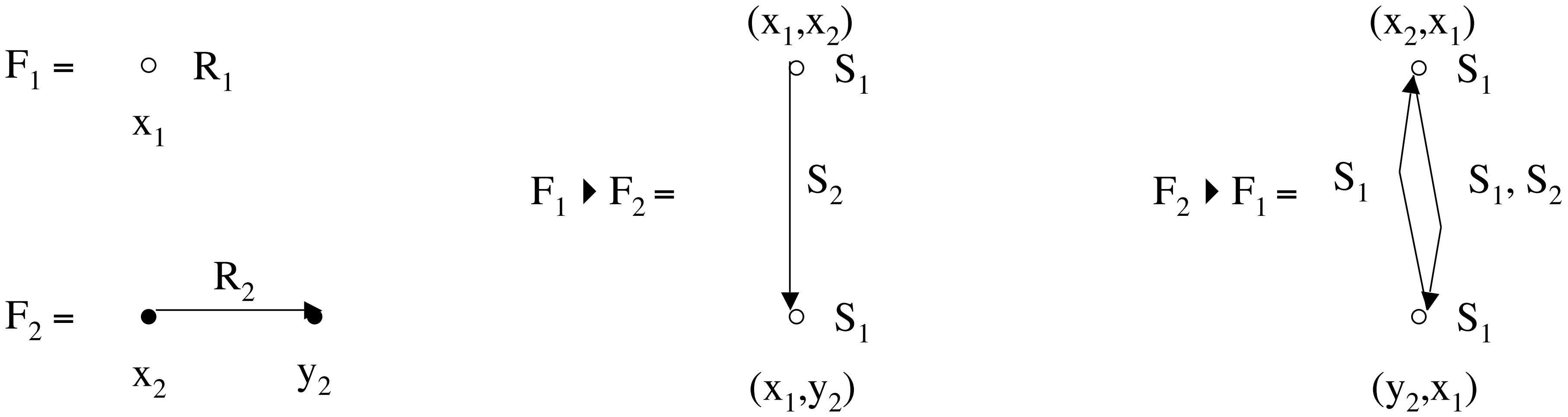
Products of relational structures

- Lexicographic products of relational structures



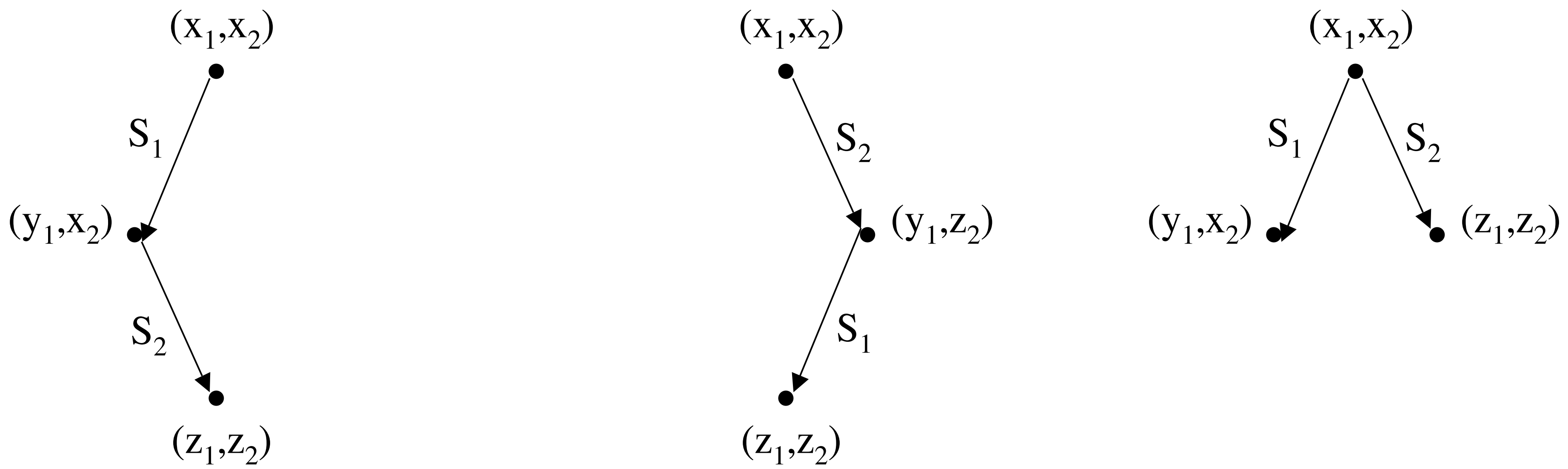
Products of relational structures

- Lexicographic products of relational structures



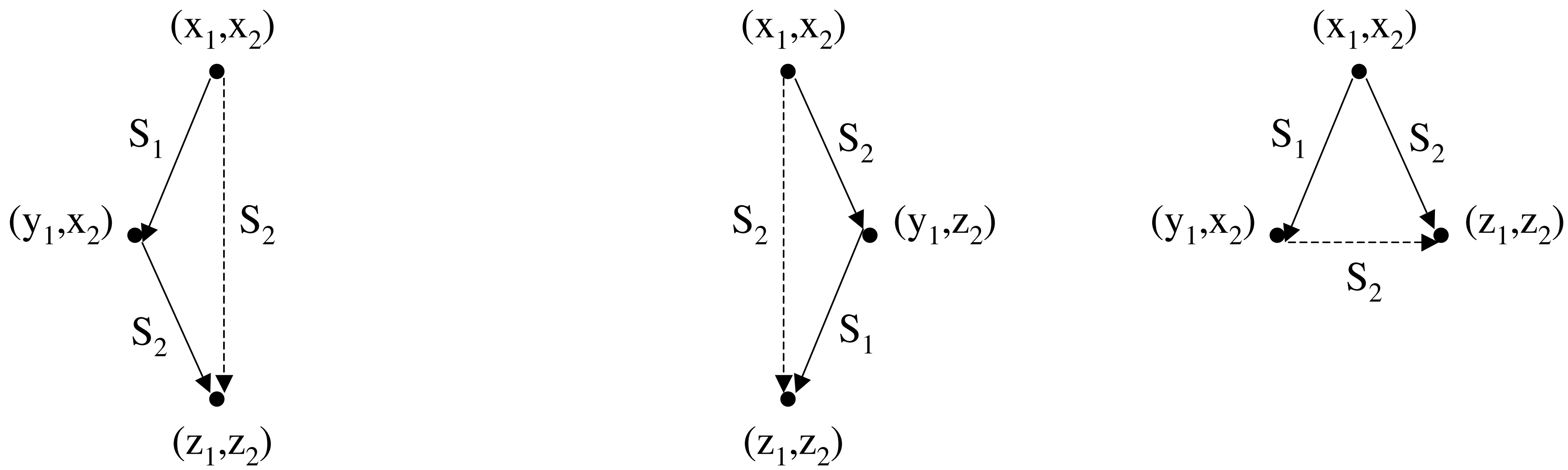
Products of relational structures

- Lexicographic products of relational structures
  - $F_1 = (W_1,R_1)$, $F_2 = (W_2,R_2)$, $F_1 \blacktriangleright F_2 = (W,S_1,S_2)$



Products of relational structures

- Lexicographic products of relational structures
  - $F_1 = (W_1,R_1)$, $F_2 = (W_2,R_2)$, $F_1 \blacktriangleright F_2 = (W,S_1,S_2)$



Products of relational structures

- Lexicographic products of relational structures
  - Let $F = (W,S_1,S_2)$ be countable, reflexive and such that
    - $\forall x \forall y(\exists z(xS_1z \ \& \ zS_2y) \Rightarrow xS_2y)$
    - $\forall x \forall y(\exists z(xS_2z \ \& \ zS_1y) \Rightarrow xS_2y)$
    - $\forall x \forall y(\exists z(zS_1x \ \& \ zS_2y) \Rightarrow xS_2y)$
  - Then there exists $F_1 = (W_1,R_1)$ and $F_2 = (W_2,R_2)$ such that F is a p-morphic image of $F_1 \blacktriangleright F_2$

Products of modal logics

- Asynchronous products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in [1] and [2] respectively
  - $L_1 \times L_2 = \mathrm{Log}\ \{ F_1 \times F_2 : F_1 \models L_1 \text{ and } F_2 \models L_2 \}$
  - $L_1 \times L_2$ is the modal logic in [1] and [2] characterized by the class of all frames of the form $F_1 \times F_2$ where $F_1 \models L_1$ and $F_2 \models L_2$

Products of modal logics

- Asynchronous products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in [1] and [2] respectively
  - L_1 and L_2 are ×-product matching iff
    - $L_1 \times L_2 = (L_1 \otimes L_2) \quad \oplus [2][1]p \to [1][2]p$

      $\oplus [1][2]p \to [2][1]p$

      $\oplus \langle 1 \rangle [2]p \to [2] \langle 1 \rangle p$

Products of modal logics

- Asynchronous products of modal logics
  - Let L_1 and L_2 be modal logics from the following list : K, D, T, K4, D4, S4, K45, KD45, S5
    - Then L_1 and L_2 are $\times$-product matching

Products of modal logics

- Lexicographic products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in [1] and [2] respectively
  - $L_1 \blacktriangleright L_2 = \mathrm{Log}\ \{ F_1 \blacktriangleright F_2 : F_1 \models L_1 \text{ and } F_2 \models L_2 \}$
  - $L_1 \blacktriangleright L_2$ is the modal logic in [1] and [2] characterized by the class of all frames of the form $F_1 \blacktriangleright F_2$ where $F_1 \models L_1$ and $F_2 \models L_2$

Products of modal logics

- Lexicographic products of modal logics
  - Let L_1 and L_2 be Kripke-complete modal logics in [1] and [2] respectively
  - L_1 and L_2 are $\blacktriangleright$-product matching iff
    - $L_1 \blacktriangleright L_2 = (L_1 \otimes L_2) \oplus [2]p \rightarrow [1][2]p$
      $\oplus [2]p \rightarrow [2][1]p$
      $\oplus \langle 1 \rangle [2]p \rightarrow [2]p$

Products of modal logics

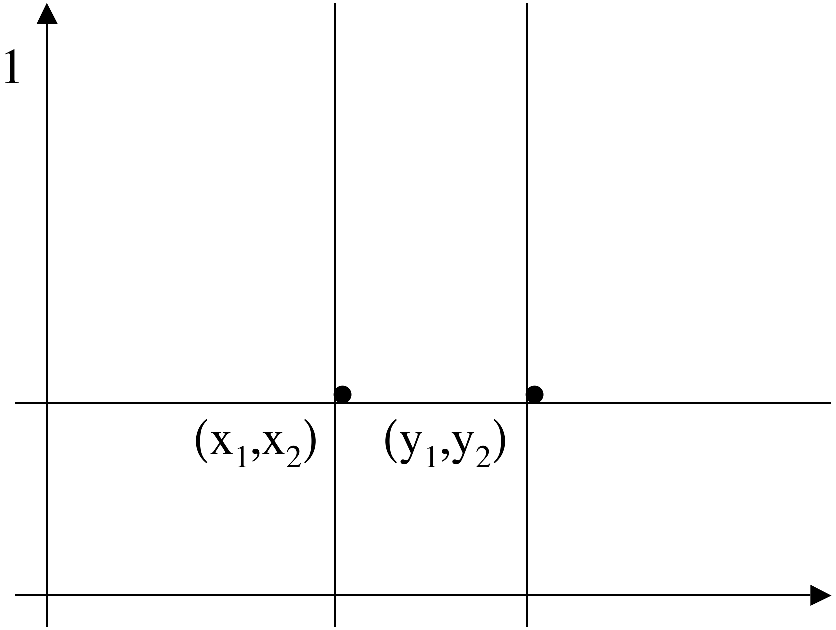
- Lexicographic products of modal logics
  - Let L_1 and L_2 be modal logics from the following list : T, B, S4, S5
    - Then L_1 and L_2 are ▸-product matching
  - Let L_2 be a modal logic from the following list : K, KB, K4, KB4
    - Then S5 and L_2 are ▸-product matching
  - Let L_1 be a canonical modal logic
    - Then L_1 and S5 are ▸-product matching

Modal logics with linear frames

- Asynchronous products of linear frames
  - $F_1 = (T,<)$, $F_2 = (T,<)$
  - $F_1 \times F_2 = (W,S_1,S_2)$ where
    - $W = T \times T$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1 < y_1$ and $x_2 = y_2$: « (x_1,x_2) is to the west of (y_1,y_2) »
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_1 = y_1$ and $x_2 < y_2$: « (x_1,x_2) is to the south of (y_1,y_2) »

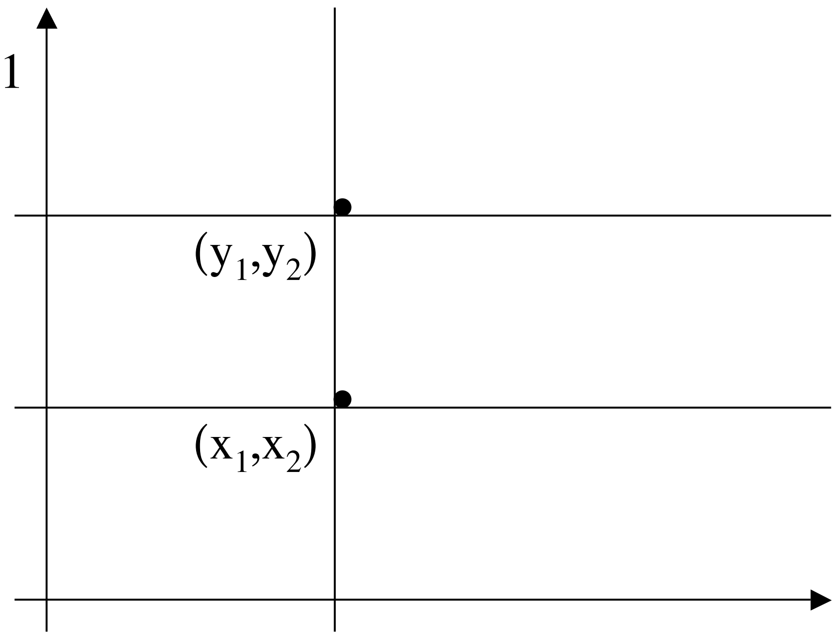
Modal logics with linear frames

- Asynchronous products of linear frames
  - $F_1 = (T,<)$, $F_2 = (T,<)$
  - $F_1 \times F_2 = (W, S_1, S_2)$
  - Let $\equiv_1$ be $S_1 \cup Id \cup S_1^{-1}$
  - $(x_1,x_2)\ S_1\ (y_1,y_2)$



Modal logics with linear frames

- Asynchronous products of linear frames
  - $F_1 = (T,<)$, $F_2 = (T,<)$
  - $F_1 \times F_2 = (W, S_1, S_2)$
  - Let $\equiv_2$ be $S_2 \cup Id \cup S_2^{-1}$
  - $(x_1,x_2)\ S_2\ (y_1,y_2)$



Modal logics with linear frames

- Asynchronous product $(Re,<)\times(Re,<)$
  - $F_1 = (Re,<)$, $F_2 = (Re,<)$
  - $F_1 \times F_2 = (W,S_1,S_2)$
    - (W,S_1) is a dense linear order without endpoints
    - (W,S_2) is a dense linear order without endpoints
    - $\forall x \forall y(\ \exists z(\ xS_1z\ \&\ zS_2y\ ) \Rightarrow \exists z(\ xS_2z\ \&\ zS_1y\ )\ )$
    - $\forall x \forall y(\ \exists z(\ xS_2z\ \&\ zS_1y\ ) \Rightarrow \exists z(\ xS_1z\ \&\ zS_2y\ )\ )$
    - $\forall x \forall y(\ \exists z(\ zS_1x\ \&\ zS_2y\ ) \Rightarrow \exists z(\ xS_2z\ \&\ yS_1z\ )\ )$
    - $\forall x \forall y \exists z(\ x \equiv_1 z\ \&\ z \equiv_2 y\ )$
    - $\forall x \forall y(\ x \equiv_1 y\ \&\ x \equiv_2 y \Rightarrow x = y\ )$

Modal logics with linear frames

- Lexicographic products of linear frames
  - $F_1 = (T,<)$, $F_2 = (T,<)$
  - $F_1 \blacktriangleright F_2 = (W,S_1,S_2)$ where
    - $W = T \times T$
    - $(x_1,x_2)\ S_1\ (y_1,y_2)$ iff $x_1 < y_1$ and $x_2 = y_2$: « (x_1,x_2) is to the west of (y_1,y_2) »
    - $(x_1,x_2)\ S_2\ (y_1,y_2)$ iff $x_2 < y_2$: « (x_1,x_2) is to the south-west, the south or the south-east of (y_1,y_2) »

Modal logics with linear frames

- Lexicographic products of linear frames
  - $F_1 = (T,<)$, $F_2 = (T,<)$
  - $F_1 \blacktriangleright F_2 = (W,S_1,S_2)$
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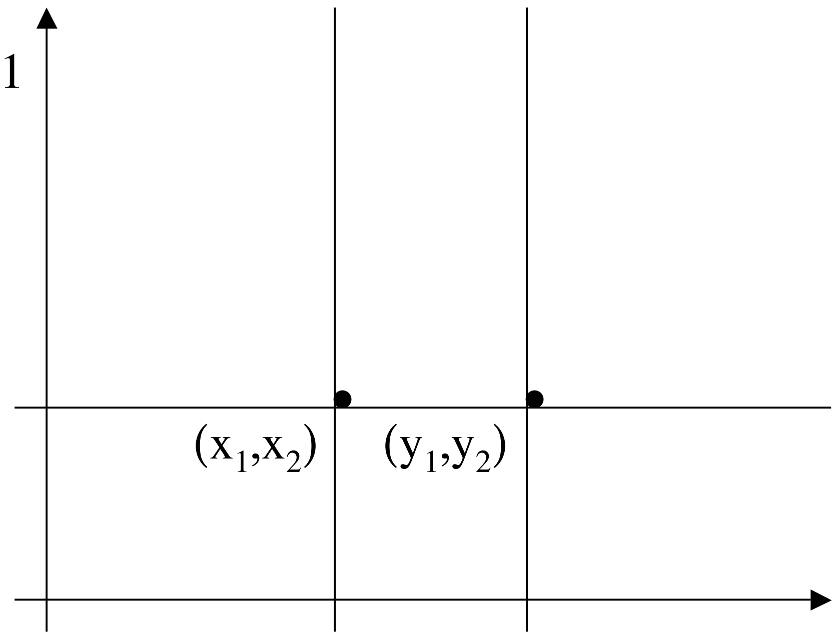
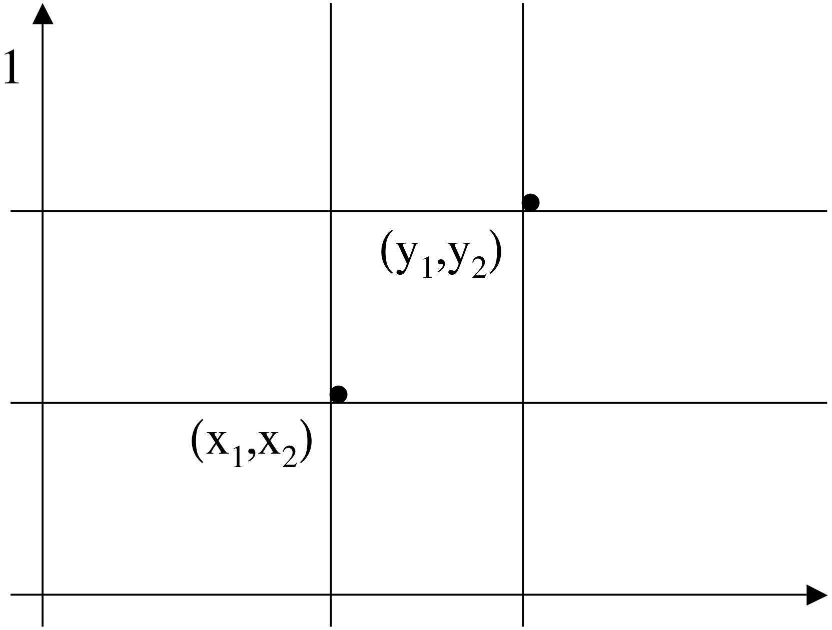


Modal logics with linear frames

- Lexicographic products of linear frames
  - $F_1 = (T,<)$, $F_2 = (T,<)$
  - $F_1 \blacktriangleright F_2 = (W,S_1,S_2)$
  - Let $\equiv_2$ be $S_2 \cup \equiv_1 \cup S_2^{-1}$
  - $(x_1,x_2)\ S_2\ (y_1,y_2)$



Modal logics with linear frames

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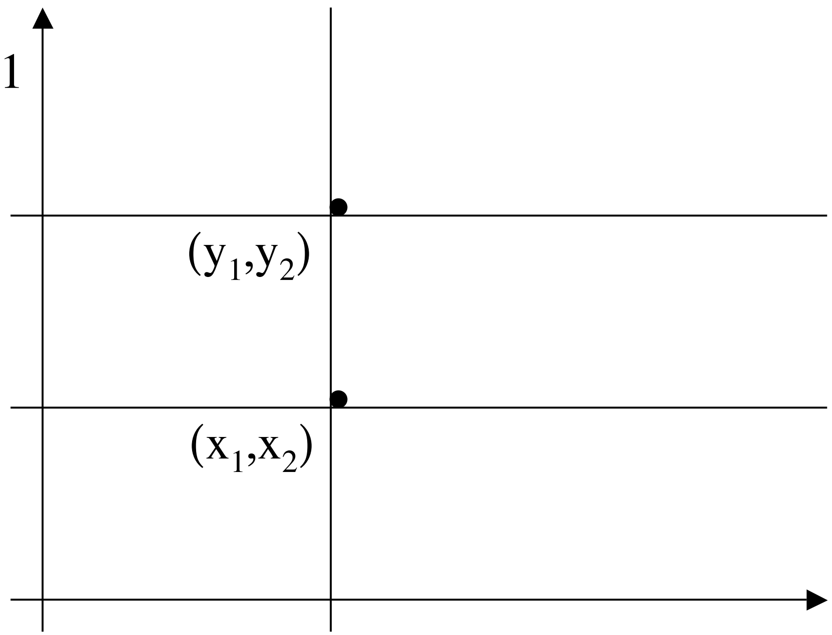
  - $(x_1,x_2)\ S_2\ (y_1,y_2)$



Modal logics with linear frames

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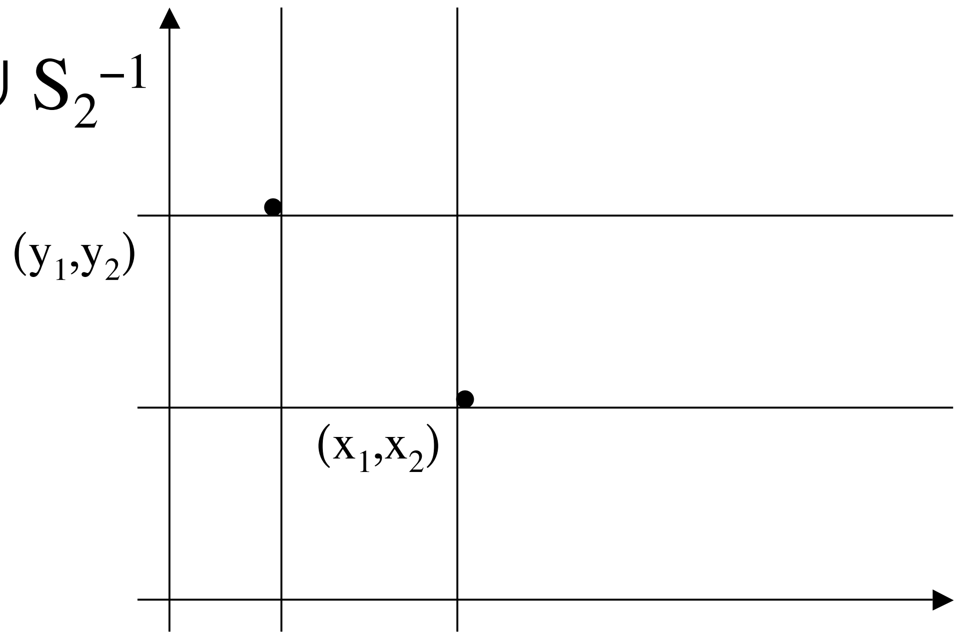

Modal logics with linear frames

- Lexicographic product (Re,<) ▸ (Re,<)
  - F_1 = (Re,<), F_2 = (Re,<)
  - $F_1 \blacktriangleright F_2 = (W, S_1, S_2)$
    - (W,S_1) is a dense linear order without endpoints
    - (W,S_2) is a dense linear (modulo $\equiv_1$) order without endpoints
    - $\forall x \forall y(\exists z(xS_1z \,\&\, zS_2y) \Rightarrow xS_2y)$
    - $\forall x \forall y(\exists z(xS_2z \,\&\, zS_1y) \Rightarrow xS_2y)$
    - $\forall x \forall y(\exists z(zS_1x \,\&\, zS_2y) \Rightarrow xS_2y)$
    - $\forall x \forall y(x \equiv_2 y)$

Modal logics with linear frames

- Asynchronous products of modal logics
  - Log $\{ (Re,<)\times(Re,<) \}$ is not r.e.
  - Log $\{ (Re,\leq)\times(Re,\leq) \}$ is not r.e.
- Lexicographic products of modal logics
  - Log $\{ (Re,<) \blacktriangleright (Re,<) \}$ is decidable (PSPACE-complete)
  - Is Log $\{ (Re,\leq) \blacktriangleright (Re,\leq) \}$ decidable ?

Modal logics with linear frames

- Asynchronous products of modal logics
  - K4.3 and K4.3 are not $\times$-product matching
  - K4.3 and S4.3 are not $\times$-product matching
  - S4.3 and K4.3 are not $\times$-product matching
  - Are S4.3 and S4.3 $\times$-product matching ?
- Lexicographic products of modal logics
  - K4.3 and K4.3 are not $\blacktriangleright$-product matching
  - Are K4.3 and S4.3 $\blacktriangleright$-product matching ?
  - S4.3 and K4.3 are not $\blacktriangleright$-product matching
  - S4.3 and S4.3 are $\blacktriangleright$-product matching

Modal logics with linear frames

- Asynchronous products of modal logics
  - K4.3×K4.3 is not decidable
  - K4.3×S4.3 is not decidable
  - S4.3×K4.3 is not decidable
  - S4.3×S4.3 is not decidable
- Lexicographic products of modal logics
  - Is K4.3 ▸ K4.3 decidable ?
  - Is K4.3 ▸ S4.3 decidable ?
  - Is S4.3 ▸ K4.3 decidable ?
  - S4.3 ▸ S4.3 is decidable (PSPACE-complete)

Open problems

- Is Log { (Re,≤) ▸ (Re,≤) } decidable ?
- Are K4.3 and S4.3 ▸ -product matching ?
- Is K4.3 ▸ K4.3 decidable ?
- Is K4.3 ▸ S4.3 decidable ?
- Is S4.3 ▸ K4.3 decidable ?
- Equivalent of Bull's theorem and Fine's theorem
  - Does every normal modal logic extending S4.3 ▸ S4.3 possess the finite model property ?
  - Is every normal modal logic extending S4.3 ▸ S4.3 finitely axiomatizable ?