Changing the order of summation for series beyond ω

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joint work with

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We have all seen tricks like this one:

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots =$$

$$\begin{array}{cccccccccc} = & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \cdots & + \\ + & & & \frac{1}{4} & + & \frac{1}{8} & + & \cdots & + \\ + & & & & & \frac{1}{8} & + & \cdots & + \\ + & & & & & & & \ddots & = \end{array}$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2\ .$$

What makes them work?

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+ & & & \frac{1}{4} & + & \frac{1}{8} & + & \cdots & + \\
+ & & & & & \frac{1}{8} & + & \cdots & + \\
+ & & & & & & & \ddots & =
\end{array}$$

$$=1+\frac{1}{2}+\frac{1}{4}+\cdots=2\,.$$

What makes them work? We have also seen these:

$$1=1+1+(-1)+1+(-1)+\cdots=2+(-1)+1+(-1)+1+\cdots=2$$

What makes them fail?

We have theorems, saying that if a series contains only nonnegative terms, or more generally, is absolutely convergent, then we can:

- reorder the terms (more precisely, subject the terms' indices to some permutation of ω)
- split the series into two according to some criteria, sum each one separately, and add the results
- split each term as a row in a $\omega \times \omega$ matrix, then sum each column, and sum the resulting series of results
- …

and we'll get the same result as when we sum the series ordinarily (seek the limit of its partial sums).

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Are all these theorems special cases of some general result?

Yes!

The trick is to generalize the notion of "reordering". Usually, it means that the sequence is composed with (ie, the indices are subjected to) some bijection from ω to ω.

To get the general result, we should consider bijections from ω to a general ordinal α. For example, three operations on the previous slide are the cases where α is ω, $\omega \cdot 2$ and ω^2 respectively.

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To get the general result, we should consider bijections from ω to a general ordinal α. For example, three operations on the previous slide are the cases where α is ω, $\omega \cdot 2$ and ω^2 respectively.

Of course, α cannot be arbitrary: bijections with ω must exist, so α must be countable. However, it is the *only* necessary condition: for every $\alpha \in \omega_1 \setminus \omega$, we can reorder the terms into an α-sequence, and we'll show that by summing it we get the same result, assuming absolute convergence.

To do that, we must first explain what does it mean to sum a hypersequence.

Hyperseries

Definition

Let $\alpha \in \mathbf{On}$ and let $(a_i : i \in \alpha)$ be a sequence of reals. *Hyperseries sum* $\sum\limits_{i\in\alpha} a_i$ is defined inductively as follows:

$$\sum_{i\in\alpha} a_i := \begin{cases} 0 & \text{, for } \alpha = 0; \\ \sum\limits_{i\in\beta} a_i + a_\beta & \text{, for } \alpha = \beta + 1; \\ \lim\limits_{\beta\in\alpha} \sum_{i\in\beta} a_i & \text{, for } \alpha \in \mathbf{Lim}. \end{cases}$$

If all the limits in the above exist, we say that the hyperseries is *convergent*.

Here, $\lim$ means the usual limit with respect to standard topology on the reals, and order topology on ordinals, that is

$$L = \lim_{\beta\in\alpha} s_\beta :\iff (\forall\varepsilon > 0)(\exists\beta_0 \in \alpha)(\forall\beta \in \alpha \setminus \beta_0)\big(|s_\beta - L| < \varepsilon\big).$$

Definition

More formally, a hyperseries is an ordered pair

$$\left((a_i)_{i\in\alpha}, (s_j)_{j\in\beta}\right),$$

where α and β are ordinals, a_i and s_j are all real numbers, and both of the following hold:

1. for every $j \in \beta$, either:
    - $j = 0$ and $s_j = 0$,
    - $j = k + 1$ and $s_j = s_k + a_k$, for some k, or
    - $j \in \mathbf{Lim}$ and $s_j = \lim_{\gamma\in j} s_\gamma$;
2. and, either
    - $\beta = \alpha + 1$ (the hyperseries *converges*), or
    - $\beta \in \mathbf{Lim}$, and $\lim_{j\in\beta} s_j$ doesn't exist in $\mathbb{R}$ (the hyperseries *diverges*).

It is easily seen that to every hypersequence $(a_i)_{i \in \alpha}$, there corresponds a unique β and sequence $(s_j)_{j \in \beta}$ such that the above holds, so we are justified in denoting a hyperseries just by $\sum(a_i)_{i \in \alpha}$.

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If the hyperseries $\sum(a_i)_{i\in\alpha} = \big((a_i)_{i\in\alpha}, (s_j)_{j\in\alpha+1}\big)$ converges, s_α is called its *sum*, and denoted by $\sum_{i\in\alpha} a_i$.

Theorem

Let $\sum_i a_i$ be an absolutely convergent series of real numbers, and let $\alpha \in \omega_1 \setminus \omega$. If $f : \alpha \to \omega$ is a bijection, then the hyperseries $\sum\limits_{\alpha} a_{f(i)}$ is convergent, and its sum is

$$\sum_{i \in \alpha} a_{f(i)} = \sum_{i=0}^{\infty} a_i \, .$$

Main theorem

Theorem 1

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Proof (transfinite induction on $\alpha \geqslant \omega$).

If $\alpha = \omega$, it is a basic result of real analysis. Hyperseries is then just a series, and an ordinary reordering takes place.
Now let $\alpha > \omega$ and let $f : \alpha \to \omega$ be a bijection. Suppose that the theorem holds for all $\beta \in \alpha \setminus \omega$.

Continuation (1)

case $\alpha = \beta + 1$

Define a sequence $(\hat{a}_i : i \in \omega)$; $\quad \hat{a}_n := \begin{cases} a_n & \text{, if } n < f(\beta); \\ a_{n+1} & \text{, if } n \geqslant f(\beta). \end{cases}$

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Function

$$g : \beta \to \omega\,; \quad g(\gamma) := \begin{cases} f(\gamma) & \text{, if } f(\gamma) < f(\beta); \\ f(\gamma) - 1 & \text{, if } f(\gamma) > f(\beta). \end{cases}$$

is a bijection, such that $a_{f(i)} = \hat{a}_{g(i)}$ for all $i \in \beta$.

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$$\sum_{i \in \alpha} a_{f(i)} = \sum_{i \in \beta} a_{f(i)} + a_{f(\beta)} = \sum_{i \in \beta} \hat{a}_{g(i)} + a_{f(\beta)} = \sum_{i=0}^{\infty} \hat{a}_i + a_{f(\beta)} = \sum_{i=0}^{\infty} a_i$$

Continuation (2)

case $\alpha \in \mathbf{Lim}$

For $\beta \in \alpha \setminus \omega$ define sequences $(a_i^\beta : i \in \omega)$ and $(\overline{a}_i^\beta : i \in \omega)$:

$$a_n^\beta := a_{\min(f[\beta] \setminus S_n)}\,; \quad \overline{a}_n^\beta := a_{\min(f[\alpha\setminus\beta] \setminus \overline{S}_n)}$$

where S_n is a set of first n elements of $f[\beta]$, while $\overline{S}_n$ is a set of first n elements of $f[\alpha \setminus \beta]$.

For these sequences,

$$\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} a_i^\beta + \sum_{i=0}^{\infty} \overline{a}_i^\beta$$

Continuation (3)

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Now we need a bijection $g : \beta \to \omega$ such that $a_{f(i)} = a^{\beta}_{g(i)}$ for all $i \in \beta$.

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$$\sum_{i \in \beta} a_{f(i)} = \sum_{i \in \beta} a^{\beta}_{g(i)} = \sum_{i=0}^{\infty} a^{\beta}_{i} = \sum_{i=0}^{\infty} a_i - \sum_{i=0}^{\infty} \overline{a}^{\beta}_{i}$$

Continuation (4)

case $\alpha \in \mathbf{Lim}$

Let $\varepsilon > 0$. We search for $\beta_0 \in \alpha$ such that for all $\beta \in \alpha \setminus \beta_0$,

$$\left| \sum_{i \in \beta} a_{f(i)} - \sum_{i=0}^{\infty} a_i \right| < \varepsilon \ .$$

Continuation (4)

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Let $\varepsilon > 0$. We search for $\beta_0 \in \alpha$ such that for all $\beta \in \alpha \setminus \beta_0$,

$$\left| \sum_{i \in \beta} a_{f(i)} - \sum_{i=0}^{\infty} a_i \right| < \varepsilon \ .$$

We know that there exists $n_0 \in \omega$ such that

$$(\forall n \in \omega \setminus n_0) \left(\sum_{i=n}^{\infty} a_i^+ < \frac{\varepsilon}{2} \bigwedge \sum_{i=n}^{\infty} a_i^- < \frac{\varepsilon}{2} \right) ,$$

where $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$.

Continuation (5)

case $\alpha \in \mathbf{Lim}$

Now we take:

$$\beta_0 := \min\left\{\gamma \in \alpha \mid \gamma > \omega \wedge (\forall \gamma' \in \alpha)(\gamma' \geqslant \gamma \rightarrow f(\gamma') > n_0)\right\}$$

$$= \max\left\{f^{-1}(0), f^{-1}(1), \ldots, f^{-1}(n_0), \omega\right\} + 1 < \alpha \ .$$

Continuation (5)

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$$= \max\left\{f^{-1}(0), f^{-1}(1), \ldots, f^{-1}(n_0), \omega\right\} + 1 < \alpha \,.$$

For all $\beta \in \alpha \setminus \beta_0$, we have

$$\left|\sum_{i \in \beta} a_{f(i)} - \sum_{i=0}^{\infty} a_i\right| = \left|\sum_{i=0}^{\infty} \overline{a}_i^{\beta}\right| = \left|\sum_{i=0}^{\infty} \left(\overline{a}_i^{\beta}\right)^{+} - \sum_{i=0}^{\infty} \left(\overline{a}_i^{\beta}\right)^{-}\right| \leqslant$$

$$\leqslant \sum_{i=0}^{\infty} \left(\overline{a}_i^{\beta}\right)^{+} + \sum_{i=0}^{\infty} \left(\overline{a}_i^{\beta}\right)^{-} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,.$$

Q.E.D.

General hyperseries

Theorem 2

Definition

We say that the hyperseries $\sum(a_i)_{i\in\alpha}$ is *absolutely convergent* if a hyperseries $\sum(|a_i|)_{i\in\alpha}$ converges.

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Theorem

Let $\alpha \in \omega_1 \setminus \omega$ and let $\sum(a_i)_{i\in\alpha}$ be an absolutely convergent hyperseries. Then $\sum(a_i)_{i\in\alpha}$ is also a convergent hyperseries. If $f : \omega \to \alpha$ is a bijection, then $\sum_i a_{f(i)}$ is an absolutely convergent series, and

$$\sum_{i=0}^{\infty} a_{f(i)} = \sum_{i\in\alpha} a_i\,.$$

General hyperseries

Proof of Theorem 2.

Let $n \in \omega$ be arbitrary, then $|a_{f(0)}| + |a_{f(1)}| + \cdots + |a_{f(n)}|$ is surely less than the fixed number $\sum_{i \in \alpha} |a_i|$, so $\sum_i |a_i|$ is a series with nonnegative terms and with partial sums bounded above, and we know that it converges.

That means $\sum_i a_i$ is absolutely convergent, and by previous theorem (using f^{-1} instead of f), we know that

$$\sum_{i \in \alpha} a_i = \sum_{i=0}^{\infty} a_{f(i)} \ .$$

$\square$

Corollary that sums(!) previous results

Corollary

Let α and β be ordinals in $\omega_1 \setminus \omega$, and let $f : \beta \to \alpha$ be a bijection between them. If $\sum(a_i)_{i\in\alpha}$ is an absolutely convergent hyperseries, then the hyperseries $\sum(a_{f(i)})_{i\in\beta}$ is also absolutely convergent, and

$$\sum_{i\in\beta} a_{f(i)} = \sum_{i\in\alpha} a_i \, .$$

Further research

- We used the results from the “ordinary” analysis, corresponding to cases ω, $\omega + 1$ and $\omega \cdot 2$ of our result. They (especially this last one) have rather convoluted proofs. Can they be streamlined into our general case? (Probably yes.)
- At one point, we split the series into positive and negative terms. What about series of complex numbers? (No problem — we split them in four quadrants.) What about series in general Banach spaces? Are norm and completeness enough? (Yes. Answered a few days ago. Still needs to be pretty-written and proof-read.)
- We sum over $\omega = \tau(\mathbb{N})$. What about summing over order types (of countable totally ordered sets) that are not ordinals? It is pretty easy to generalize the definition and the result to sums over $\pi = \tau(\mathbb{Z})$. What about $\eta = \tau(\mathbb{Q})$?

Питана?