Complex algebras of natural numbers

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Overview

1. Definitions of algebras of arithmetic circuits.
2. Circuit definable sets and functions.
3. Some algebraic properties.
4. Decidability questions.

The structures

$$\begin{array}{ll} \mathbb{N} := \langle \omega, +, 0, \cdot, 1 \rangle, & \mathfrak{Cm}(\mathbb{N}) := \langle 2^\omega, \cup, \cap, \emptyset, \omega, +, \{0\}, \bullet, \{1\} \rangle \\ \mathbb{N}^+ = \langle \omega, +, 0 \rangle, & \mathfrak{Cm}(\mathbb{N})^+ := \langle 2^\omega, \cup, \cap, \emptyset, \omega, +, \{0\} \rangle \\ \mathbb{N}^\cdot = \langle \omega, \cdot, 1 \rangle & \mathfrak{Cm}(\mathbb{N})^\bullet := \langle 2^\omega, \cup, \cap, \emptyset, \omega, \bullet, \{1\} \rangle \end{array}$$

$+$	$a + b$	$= \{n + m : n \in a,\ m \in b\}.$
$\cdot$	$a \bullet b$	$= \{n \cdot m : n \in a,\ m \in b\}.$
$\geq$	$\uparrow a$	$= \{k : (\exists n)[n \in a \text{ and } n \leq k\}.$
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If $\mathfrak{A}$ is an atomic Boolean algebra with operators, we set

$$\mathfrak{A}_0 := \text{Subalgebra of } \mathfrak{A} \text{ generated by the constants,}$$

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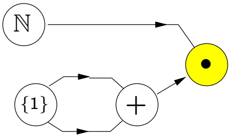
The elements of $\mathfrak{Cm}(\mathbb{N})_0$ are called *arithmetic circuits* (McKenzie, Wagner, 2003,2007).

Arithmetic circuits can be displayed graphically

- $(\{1\} + \{1\}) \bullet \omega$, can be depicted as



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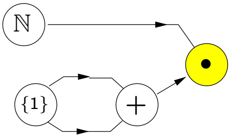
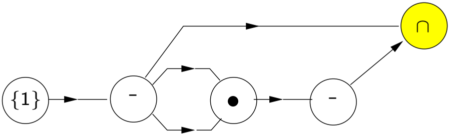


- $\overline{\{1\}} \cap \overline{(\overline{\{1\}} \bullet \overline{\{1\}})}$ can be depicted as



- Some well-known mathematical conjectures can be 'expressed' in terms of arithmetic circuits.
- Consider the circuit

$$\tau_{\mathrm{g}} = \tau_{\mathrm{e}} \cap \overline{(\{0\} \cup (\{1\} + \{1\}) \cup (\tau_{\mathrm{p}} + \tau_{\mathrm{p}}))}$$

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  where τ_{e} is the circuit defining the even numbers, and τ_{p} the circuit defining the primes.
- The set τ_{g} is empty if and only if Goldbach’s conjecture is true.

Functions

- Any circuit τ featuring variables $x_1, \ldots, x_k$ defines a function $\tau(x_1, \ldots x_k) : (2^\omega)^k \longrightarrow 2^\omega$ in the obvious way.
- Some circuit–definable functions:
  - The circuit $\tau_{\mathrm{u}}(x) = x + \omega$ defines the 'up'-function:

$$\uparrow x = \{n \in \omega : (\exists m \in x), m \leq n\}$$

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  - The circuit $\tau_{\min}(x) = (\overline{(x + \omega)} + \{1\}) \cap x$ defines the minimum function for non–empty sets.

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$$s \dot{-} t = \{k \in \omega : (\exists n, m \in \omega)[n \in s,\ m \in t,\ k = n - m]\}$$

$$F_{\max}(s) = \{\max(s)\} \quad \text{for finite, non-empty } s$$

$$F_{\mathrm{fin}}(s) = \begin{cases} \omega & \text{if } s \text{ is finite} \\ \emptyset & \text{otherwise} \end{cases}$$

Σs if s is finite

$|s|$ if s is finite.

Sets of formulas

Suppose that K is a class of algebras of the same type $\mathcal{O}$. We consider the following sets of formulas in the language of $\mathcal{O}$ (plus equality):

1. The *first-order theory* **FO** K *of* K: The set of first-order formulas in the language $\mathcal{O}$ true in all algebras in K.
2. The *equational theory* **Eq** K *of* K: The set of formulas in the language of the forms $\tau = \sigma$ whose universal closures are true in K.
3. The *satisfiable equations* **EqSat** K *of* K: The set of formulas of the forms $\tau = \sigma$ whose existential closures are true in each member of K.

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- **Var** $\mathfrak{Cm}(\mathbb{N}^+)$ is generated by countably many finite finitely based subdirectly irreducible algebras.

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- **Var** $\mathfrak{Cm}(\mathbb{N}^+)$ is generated by countably many finite finitely based subdirectly irreducible algebras.
- **Eq** $\mathfrak{Cm}(\mathbb{N}^+)_0 =$ **Eq** $\mathfrak{Cm}(\mathbb{N}^+)$.
- **Eq** $\mathfrak{Cm}(\mathbb{N}^{+,d})_0 \neq$ **Eq** $\mathfrak{Cm}(\mathbb{N}^{+,d})$.
- **FO** $\mathfrak{Cm}(\mathbb{N}^+)_0 \neq$ **FO** $\mathfrak{Cm}(\mathbb{N}^+)$.

Decidability questions 1

Membership problem: Given an arithmetic circuit τ and a number m, determine whether $m \in \tau$.

Emptiness problem: Given an arithmetic circuit τ, determine whether $\tau = \emptyset$.

Satisfiability problem: Given an n–ary term function $\tau(x_1, \ldots, x_n)$ and some $k \in \omega$, determine whether there are $k_1, \ldots, k_n \in \omega$ such that $k \in \tau(k_1, \ldots, k_n)$.

- Problems 1 and 2 are obviously computably equivalent: if one is decidable, so is the other. It is not known whether any of these problems is decidable.

- Variable-free arithmetic circuits without the $\bullet$-gate are called *integer expressions*.
- The membership and non-emptiness problems for integer expressions are $\mathrm{PS}\textsc{pace}$-complete (Stockmeyer and Meyer 1973).
- Complexity-theortic results for various collections of gates can be found in (McKenzie and Wagner 2003, 2007), (Yang 2000) (Glaßer *et al.* 2007, 2007)
- For results on equations involving integer expressions, see (Jeż and Okhotin 2008, 2008).

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Thank you
Dziękuję
Asante
Danke
Merci

- The recognition complexity for every *fixed* circuit-definable set is relatively low.

Theorem

Every circuit-definable set is in the bounded arithmetic hierarchy, BA (and hence its characteristic function is in $\mathcal{E}^0_*$).

- Hence, every circuit-definable set is certainly:
  - in the polynomial hierarchy, PH;
  - in $\textsc{DSpace}(n) = \mathcal{E}^2_*$.
- All circuit-definable sets are certainly context-sensitive. However, the set of primes, which is circuit-definable, is not context-free (Hartmanis and Shank 1968).
- Theorem 1 notwithstanding, no nice examples of non-circuit-definable sets are known!

- First main result: functions from $\mathbb{N}$ to $\mathbb{N}$ having (roughly speaking) infinite range and sublinear growth are not circuit-definable:

Theorem

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. If the set

$$\{f(n) : n \in \mathbb{N},\ f(n) < n\}$$

is infinite, then f is not circuit-definable.

- Second main result: functions from $2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ which (roughly speaking) have finite range and fail to converge on certain 'sparse' chains under inclusion are not circuit-definable.

Definition

Let s be a finite, non-empty set of numbers, t a set of numbers, and m a number. We write $s \sqsubseteq_m t$ if $m \geq \max(s)$ and $s = t \cap \{i \mid i \leq m\}$.

Theorem

Let $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be a function with finite range. And suppose that, for all finite, non-empty $s \subseteq \mathbb{N}$ and all $m \geq \max(s)$, there exists $t \subseteq \mathbb{N}$ for which $s \sqsubseteq_m t$ and $F(t) \neq F(s)$. Then F is not circuit-definable.