Undecidability of the Theory of Projective Planes

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Definition of Projective Plane

A *projective plane* is a structure $\langle A, (A^0, {}^0A), \cdot \rangle$ with a disjunction of A into two subsets $A^0 \cup {}^0A = A$, $A^0 \cap {}^0A = \emptyset$ and commutative partial operation "$\cdot$" which satisfy the following properties:

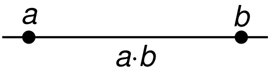
(1) $a \cdot b$ is defined iff $a \neq b$ and $a, b \in A^0$ (or $a, b \in {}^0A$) with the product $a \cdot b \in {}^0A$ ($a \cdot b \in A^0$ respectively);

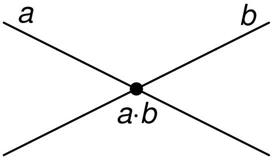
(2) for all $a, b, c \in A$ if $a \cdot b$, $a \cdot c$, $(a \cdot b) \cdot (a \cdot c)$ are defined, then $(a \cdot b) \cdot (a \cdot c) = a$;

(3) there exist distinct $a, b, c, d \in A$ such that products $a \cdot b$, $b \cdot c$, $c \cdot d$, $d \cdot a$ are defined and pairwise distinct.

Geometric Sense

One may think of elements of A^0 as "points" and elements of ${}^0A$ as "lines".





Models for Projective Planes

In any projective plane $\mathcal{A} = \langle A, (A^0, {}^0A), \cdot\rangle$ we replace the binary operation by its graph

$$P^{\mathcal{A}} = \{\langle a, b, c\rangle \mid a, b, c \in A,\ a{\cdot}b \downarrow= c\},$$

and consider $\mathcal{A}$ as a model of predicate signature

$$\sigma = \langle A^0, {}^0A, P\rangle.$$

This allows us to apply methods of model theory and investigate the question of decidability of elementary theories.

Similar classes

The theory is decidable:

- Abelian groups

The theory is undecidable:

- groups
- semigroups
- commutative semigroups
- projective geometries (simple sectionally complemented modular lattices of finite height)

In the present paper

We prove that the class of symmetric, irreflexive graphs is relatively elementarily definable in the class of projective planes.

This implies that the theory of all projective planes is undecidable.

Definition of Interpretation

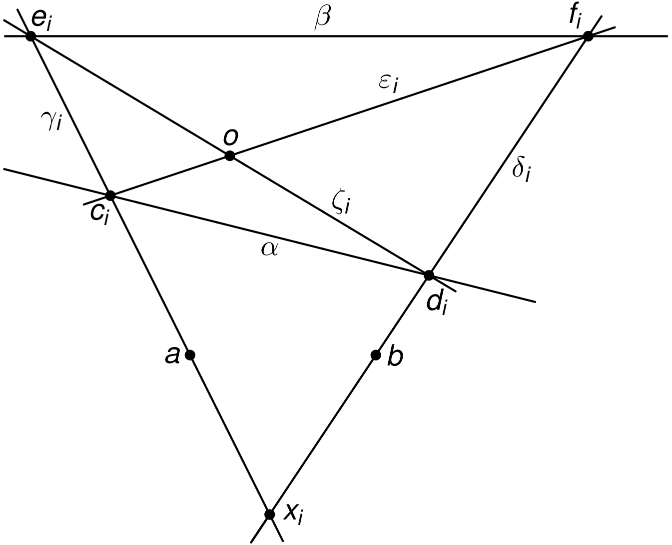
Let $\mathcal{G} = \langle G, E \rangle$ be a symmetric, irreflexive graph such that G is an initial segment of ω.

We define a nondegenerated configuration $\mathcal{A} = \langle A, (A^0, {}^0A), I \rangle$ where

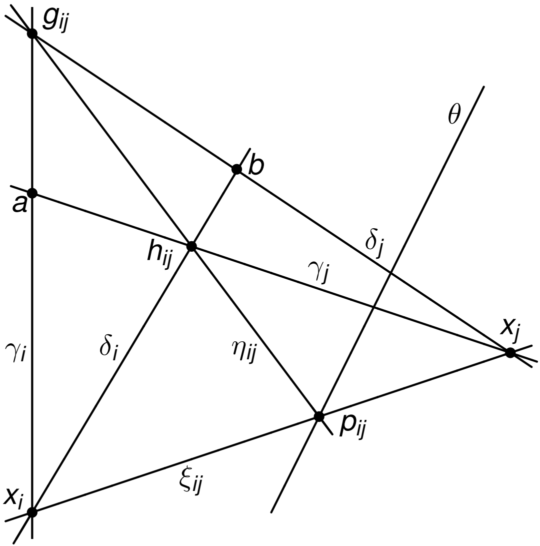
$$A^0 = \{a, b, c_i, d_i, e_i, f_i, o, x_i \mid i \in G\} \cup \{g_{ij}, h_{ij}, p_{ij} \mid i, j \in G, i < j\},$$
$${}^0A = \{\alpha, \beta, \gamma_i, \delta_i, \varepsilon_i, \zeta_i, \theta \mid i \in G\} \cup \{\eta_{ij}, \xi_{ij} \mid i, j \in G, i < j\},$$

and incidence relation I is shown in the pictures below.

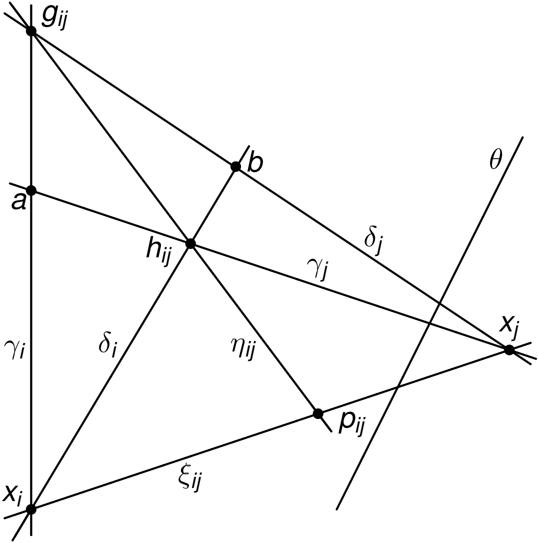
Incidence Relation (Fig.1)



Incidence Relation (Fig.2): $\langle i, j \rangle \in E$



Incidence Relation (Fig.3): $\langle i, j \rangle \notin E$



Freely Generated Projective Plane

Now consider the projective plane $\mathcal{F} = \langle F, (F^0, {}^0F), \cdot \rangle$ freely generated by the configuration $\mathcal{A} = \langle A, (A^0, {}^0A), I \rangle$, i.e. there exists a sequence $\mathcal{A} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ of configurations such that $\mathcal{F} = \bigcup_{i \in \omega} \mathcal{A}_i$ and for all $i \in \omega$ the following conditions hold

(1) for all equitype $a, b \in \mathcal{A}_i$ with $a \neq b$ there exists a $c \in \mathcal{A}_{i+1}$ such that $c = a \cdot b$;

(2) for any $c \in \mathcal{A}_{i+1} \backslash \mathcal{A}_i$ there exist only two elements $a, b \in \mathcal{A}_i$ such that $c = a \cdot b$.

Relative Elementary Definability

We construct two formulas $\Phi(w)$ and $\Psi(u, v)$ of the signature σ with parameters $a, b, o, \alpha, \beta, \theta$ and for any symmetric, irreflexive graph $\mathcal{G} = \langle G, E \rangle$ define subsets

$$G_0 = \{w \in F \mid \mathcal{F} \models \Phi(w)\},$$
$$E_0 = \{\langle u, v \rangle \in F^2 \mid \mathcal{F} \models \Psi(u, v)\}$$

such that the graph $\mathcal{G}_0 = \langle G_0, E_0 \rangle$ is isomorphic to $\mathcal{G}$.

Main Results

Since the theory of symmetric, irreflexive graphs is hereditarily undecidable, we obtain the following results:

(1) The class of all projective planes has hereditarily undecidable theory.

(2) The class of freely generated projective planes has hereditarily undecidable theory.

Open Questions

- Is the theory of all Desarguesian projective planes decidable?
- Is the theory of all Pappian projective planes decidable?
- Is the theory of an arbitrary free projective plane decidable?

References

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2. *Yu.L.Ershov, I.A.Lavrov, A.D.Taimanov, M.A.Taitslin*, Elementary Theories, Russian Mathematical Surveys, vol. 20 (1965), no. 4, 35–105.