Complexity of Linear Extensions in the Ershov Hierarchy

(Joint work with S. B. COOPER and A. MORPHETT)

KYUNG IL LEE

University of Leeds

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Preliminaries and the Scope.

- We are interested in absolute[1] descriptive[2] complexity of a **linear extension** $<_B$ $(\subset \mathbb{N} \times \mathbb{N})$ of a **partial ordering** $<_A$.
- Complexity of a linear ordering $<_B$ can be measured in linearising a partial ordering $<_A$.
- Some computable approximations could involve the following cases:
  1. $<_B$ is Σ_1.
  2. $<_B$ is Δ_2.
  3. $<_B$ is Σ_n.

  ...

  1. $<_B$ is n-c.e.
  2. $<_B$ is ω-c.e.

  ...

[1] not relative

[2] We are not interested in hierarchy or classification via oracle strength in this talk.

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Previous Works (1)

Theorem (Szprilrajn, 1930)

If $\mathcal{P}$ is a partial ordering, then $\mathcal{P}$ has a linear extension.

Theorem (Folklore)

If $\mathcal{P}$ is a computable partial ordering, then $\mathcal{P}$ has a computable linear extension.

Previous Works (2)

- Some other restrictions to preserve: (computably) well-foundedness, (computably) scatteredness, and so forth.

Definition

An ordering is ***well-founded*** if it has no infinite descending sequence.

Definition

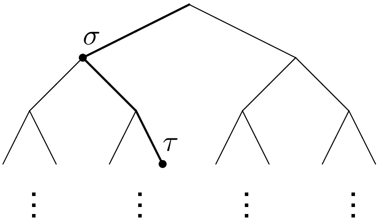
An ordering is ***computably well-founded*** if it has no infinite descending sequence which is *computable*.

Terminology

- A ***computable sequence*** is a computable function $i \mapsto x_i$ from $\mathbb{N}$ to $\mathbb{N}$.
  – If $x_0, x_1, \ldots$ is an infinite computable sequence, then $W = \{x_0, x_1, \ldots\}$ is a c.e. set.
  – Conversely, every c.e. set W_e exhaustively provides a (possibly finite) computable sequence $x_0, x_1, \ldots$ in order of enumeration.
- W_e is a ***descending sequence*** if $x_i >_B x_{i+1}$ for all $i \in \mathbb{N}$.
- Suffices to make sure $x_i <_B x_{i+1}$ for some i in order to give a computably well-foundedness.

Example

The full binary tree 2^ω is a partial ordering which is NOT computably well-founded since it does have computable path.



- Have defined an ordering: $\tau < \sigma$ if σ, τ both in $2^{<\omega}$ and $\sigma \subset \tau$.
- Can define a particular (possibly computable) partial ordering as a subtree of the full binary tree.

Previous Works (3)

Theorem (Bonnet, 1969)

Every well-founded partial ordering has a well-founded linear extension.

Theorem (Rosenstein and Kierstead, 1984)

Every well-founded computable partial ordering has a well-founded computable linear extension.

Previous Works (4)

Theorem (Rosenstein and Statman, 1984)

There is a computably well-founded computable partial ordering with no computable well-founded linear extension which is computable.

Sketch Proof. next page!

Theorem (Rosenstein, 1984)

Every computably well-founded computable partial ordering has a computably well-founded linear extension which is Δ^0_2.

Theorem (Rosenstein and Statman, 1984)

There is a computably well-founded computable partial ordering with no computable well-founded linear extension which is computable.

Sketch Proof.

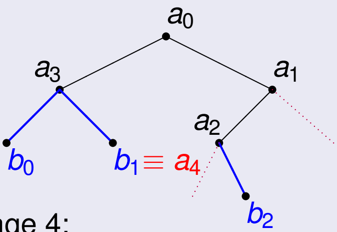
Two steps of a strategy due to Rosenstein and Statman:

1. Construct a computable binary tree (a particular computable partial ordering) with no computable path (computably well-founded).
2. Prove such partial orderings have no computably well-founded computable linear extensions.
(Need to construct suitable infinite descending computable sequences W's, say.)

Sketch Proof. (continued.)

- Assume $<_A$ is a comp. infinite tree with no comp. path.
- Assume $<_B$ is comp. lin. ext. of $<_A$.

Construction:



For example, at stage 4:

1. Have defined $W^4 = \{a_0 >_B a_1 >_B a_2 >_B a_3\}$.
2. Search for immediate predecessors of W^4 which agrees with $<_A$: $P^4 = \{b_0, b_1, b_2\}$.
3. b_0, b_1, b_2 are incomparable under $<_A$, so set $a_4 \equiv \max\{b_0, b_1, b_2\}$ under $<_B$.

Sketch Proof. (continued.)

Verification:

1. Prove $<_B$ is computable.
2. Prove $<_B$ is infinite.
3. Prove $a_n >_B a_{n+1}$ for all $n \in \mathbb{N}$.
   (W is an infinite descending sequence.)

□

Our Questions.

- Expecting negative answers, we ask:
  1. Does there exist a computably well-founded computable partial ordering which has no computably well-founded linear extension that is *d*-c.e.?
  2. $\cdots$ *n*-c.e.?
  3. $\cdots$ ω-c.e.?
- Or expecting affirmative answers, we ask:
  1. Does any computably well-founded computable partial ordering have a computably well-founded linear extension that is ω-c.e.?
  2. $\cdots$ *n*-c.e.?
  3. $\cdots$ *d*-c.e.?

Our Answers.

- Expecting negative answers, we ask:
  1. Does there exist a computably well-founded computable partial ordering which has no computably well-founded linear extension that is *d*-c.e.? (✓)
  2. $\cdots$ *n*-c.e.? (**?**)
  3. $\cdots$ ω-c.e.? ($\times$)
- Or expecting affirmative answers, we ask:
  1. Does any computably well-founded computable partial ordering have a computably well-founded linear extension that is ω-c.e.? (✓)
  2. $\cdots$ *n*-c.e.? (**?**)
  3. $\cdots$ *d*-c.e.? ($\times$)

Our Results and Progress (1)

Theorem (Work in Progress)

There is a computably well-founded computable partial ordering which has no d.c.e. linear extension which is computably well-founded.

(Approach) Involves elaborating Rosenstein's strategy, defining infinitely many descending sequences (instead of W's), corresponding to the infinite list of possible d.c.e. linear extensions.

Theorem

Every computably well-founded computable partial ordering has a computably well-founded linear extension which is ω-c.e.

$\leftarrow$ These are the main results.

Proof of the Main Result

Theorem

Every computably well-founded computable partial ordering has a computably well-founded linear extension which is ω-c.e.

Sketch Proof.

Let $x^e_0, x^e_1, \ldots \in W_e$ in order of enumeration.
Requirement: $|W_e| = \infty \Rightarrow x^e_i <_B x^e_j$ for some $i < j$.
Basic strategy:

- Find x_i, x_j ($i < j$) such that $x_i \mid_A x_j$ and set $x_i <_B x_j$. ($<_A$ is computably well-founded guarantees that such x_i, x_j exist.)
- Apply a finite injury argument.
- We can see a bound on the number of mind-changes from the construction.
- It turn out to be at most computably bounded (Our bound is 2^e for $<_{B,s}\restriction e$ via some technical parameter.) □

Our Results (2)

Theorem

Every computably well-founded Δ_2 partial ordering has a computably well-founded linear extension which is Δ_2.

Theorem (Work in Progress)

Every computably well-founded ω-c.e. partial ordering has a computably well-founded linear extension which is ω-c.e.

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