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A REVIEWED SYNTACTIC PROOF OF GÖDEL INTERPRETATION OF INTUITIONISTIC LOGIC INTO S4

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Objective

We aim to present an alternative proof of Gödel's interpretation of intuitionistic logic into S4 classical modal logic system.

- Gödel's interpretation.
- Gödel's conjecture
- Gödel's aim
- Proofs of Gödel's conjecture
- Preliminary Results
- A reviewed syntactic proof of Gödel interpretation.

Gödel's interpretation

The system Σ

- $Bp \to p$
- $Bp \to (B(p \to q) \to Bq)$
- $Bp \to BBp$
- $B\alpha$ may be inferred from α
- the axioms and inference rules of classical propositional calculus.

Translation of Heyting's calculus into Σ

Heyting's calculus	The system Σ
$\sim p$	$\neg Bp$
$(p \supset q)$	$(Bp \to Bq)$
$(p \nabla q)$	$(Bp \vee Bq)$
$(p \triangle q)$	$(p \wedge q)$

Gödel's conjecture

A formula holds in Heyting's calculus if and only if its translation is provable in Σ

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Gödel's aim

To give an interpretation of the propositional intuitionistic logic which was meaningful also from a non intuitionistic point of view.

Proofs of Gödel’s conjecture

1948: the first semantic proof

J. C. C. McKinsey and Tarski introduced three different translations of Heyting’s calculus into S4 and showed them through algebraic and topological semantics [5].

1963: the first syntactic proof

Ian Hacking showed that one of J. C. C. McKinsey and Tarski‘s interpretation could be restricted to the S3 system, using the cut elimination property of the modal logic system considered [3].

1968: a first translation general theory

Prawitz e Malmnäs outlined a first translation general theory, in which, they introduced a syntactic proof of Gödel’s conjecture using proof theory resources [8].

Now, we present one more syntactic proof to Godel’s conjecture, taking into account the recent results and approaches about S4 modal logic in natural deduction. Particularly, taking into account the structural properties of modal inference rules introduced by Medeiros in 2006.

1 Preliminary Results

- The Propositional intuitionistic calculus in natural deduction - I

$L(\mathbf{I}) = \{\rightarrow, \wedge, \vee, \bot\}$

Inference rules:

$$(I_{\rightarrow})\ \dfrac{\begin{matrix}[A]^i\\ \vdots \\ B\end{matrix}}{A \rightarrow B}\,i \qquad (E_{\rightarrow})\ \dfrac{A \qquad A \rightarrow B}{B}$$

$$(I_{\wedge})\ \dfrac{A \qquad B}{A \wedge B} \qquad (E_{\wedge})\ \dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$$

$$(I_{\vee})\ \dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B} \qquad (E_{\vee})\ \dfrac{A \vee B \quad \begin{matrix}[A]^i\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]^j\\ \vdots \\ C\end{matrix}}{C}\,i,j$$

$$(\bot_I)\ \dfrac{\bot}{A}$$

Theorem 1.1 (Normalization theorem) *If $\Gamma \vdash_I A$, then there is a normal derivation of A from Γ in* **I**.

Corollary 1.2 (Subformula principle) *Every formula that occurs in a normal derivation of A from Γ in* **I** *is a subformula of A or of some formula of Γ*

- **The S4 classical system in natural deduction - NS4**

$L(\mathbf{NS4}) = \{\rightarrow, \wedge, \vee, \bot, \Box\}$.

Inference rules: The same introduction and elimination rules for the propositional logical connectives of **I**, and:

$$(E_\Box) \qquad \frac{\Box A}{A}$$

$$(I_\Box) \qquad \frac{\Box B_1 \ldots \Box B_n \qquad \begin{array}{c}[\Box B_1]^{i_1} \ldots [\Box B_n]^{i_n} \\ \vdots \\ A\end{array}}{\Box A}\, i_1 \ldots i_n$$

$$(\bot_c) \qquad \frac{\begin{array}{c}[A \rightarrow \bot]^i \\ \vdots \\ \bot\end{array}}{A}\, i$$

Restrictions on $(I_\Box)$: all the occurrences of assumptions in the class $[\Box B_k]^{i_k}, 1 \leq k \leq n$, must be discharged by the application of the rule. Furthermore, the premise A must not depend on any assumptions other than those occurring in the assumption classes in question.

Theorem 1.3 *If* $\Gamma \vdash_{NS4} A$*, then there is a normal derivation of* A *from* Γ *in* **NS4**.

Corollary 1.4 (Subformula principle) *Every formula that occurs in a normal derivation* Π *of* C *from* Γ *in* **NS4** *is a subformula of* C *or of some formula of* Γ*, except for the assumptions discharged by applications of* $(\bot_c)$ *and for occurrences of* $\bot$ *in segments that are immediately below such assumptions.*

The proofs of 1.3 e 1.4 was made by Medeiros in [6].

- **The S4 intuitionistic system in natural deduction - IS4**

$L(\mathbf{NS4}) = \{\rightarrow, \wedge, \vee, \bot, \Box, \Diamond\}$

The inference rules:
The same of the system **I**,
The modal rules $(I_\Box)$ and $(E_\Box)$ of the **NS4**, and:

$$(I_\Diamond) \quad \frac{A}{\Diamond A}$$

$$(E_\Diamond) \quad \frac{\Box B_1 ... \Box B_n, \Diamond C \qquad \begin{matrix} [\Box B_1]^{i_1} ... [\Box B_n]^{i_n}, [C]^{i_{n+1}} \\ \vdots \\ \Diamond A \end{matrix}}{\Diamond A} \; i_1 ... i_{n+1}$$

Restrictions on $(E_\Diamond)$: all the occurrences of assumptions in the class $[\Box B_k]^{i_k}, 1 \leq k \leq n$, and in $[C]^{i_{n+1}}$ must be discharged by the rule application. Furthermore, the premise $\Diamond A$ must not depend on any assumptions other than those occurring in the assumption classes in question.

2 Translation of I into NS4

Definition 2.1 *Let g be a function that maps formulas of $L(\mathbf{I})$ into formulas of $L(\mathbf{NS4})$ defined as following:*

$$
\begin{aligned}
g(A) &= \Box A, \text{ if } A \text{ is an atomic formula} \\
g(A \to B) &= \Box(g(A) \to g(B)) \\
g(A \wedge B) &= g(A) \wedge g(B) \\
g(A \vee B) &= \Box g(A) \vee \Box g(B) \\
g(\bot) &= \bot
\end{aligned}
$$

It follows from this definition that $g(\neg A) = \Box\neg g(A)$.

Lemma 2.2 *If $\Gamma \vdash_I A$, then $g(\Gamma) \vdash_{NS4} g(A)$, for every formula $A \in L(\mathbf{I})$.*

Proof.
$\Gamma \vdash_I A$ implies $g(\Gamma) \vdash_{IS4} g(A)$,
$\vdash_{IS4} g(A) \leftrightarrow \Box g(A)$.
IS4 is a subsystem of **NS4**.

Lemma 2.3 *If* Π *is a normal derivation of* $g(A)$ *from* $g(\Gamma)$ *in* **NS4**, *then* Π *can be transformed into a normal derivation* Π' *in which no one application of* $(\bot_c)$ *has as discharged assumption a formula of the form* $\neg(A \to B)$.

Proof. Let $n(\Pi)$ be the number of applications of $\bot_c$ with discharged assumptions of form $\neg(A \to B)$.

$$\Pi \equiv \quad \dfrac{\begin{matrix}[\neg(A \to B)]^i \\ \Sigma \\ \bot\end{matrix}}{\begin{matrix}A \to B \\ \Pi_0\end{matrix}}\, r, i$$

where Σ is a subderivation of Π having no applications of $(\bot_c)$ whose discharged assumption has the form $\neg(A \to B)$.

Case 1) In Π the assumption $\neg(A \to B)$ is not a major premise of an elimination rule.

Case 2) In Π the assumption $\neg(A \to B)$ is a major premise of a elimination rule.

$$\Pi \equiv \quad \dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ A \to B\end{matrix} \qquad [\neg(A \to B)]^i}{\begin{matrix}\bot \\ \Sigma_2 \\ \bot\end{matrix}}}{\begin{matrix}A \to B \\ \Pi_0\end{matrix}}\, i$$

Lemma 2.4 *If $g(\Gamma) \vdash_{NS4} g(A)$, then $g(\Gamma) \vdash_{IS4} g(A)$, for every $A \in L(I)$*

- Let us assume $g(\Gamma) \vdash_{NS4} g(A)$.
- There is a normal derivation of $g(A)$ from $g(\Gamma)$ by the lemma 1.3
- This derivation can be transformed into a normal derivation such that no one application of $(\perp_c)$ has any discharged assumption of the form $\neg(A \rightarrow B)$, by the lemma 2.3
- Let Π be such derivation.
- We will show that Π can be transformed into a derivation without applications of $(\perp_c)$
- Let Σ be a subderivation of Π that has just one application of the $(\perp_c)$ rule and this application is the last inference rule of Σ.

$$\Pi \equiv \begin{array}{c} [\neg B]^i \\ \Sigma \\ \dfrac{\perp}{B}\, r, i \\ \Pi_0 \end{array}$$

- In Σ the assumption $\neg B$ cannot be:
  1. premise of an application of $(I_\Box)$, because this contradicts the restrictions of the rule.
  2. premise of any introduction rule nor minor premise of $(E_\rightarrow)$, since these contradict the Subformula principle.

$$\Pi \equiv \begin{array}{c} \Sigma_1 \\ \dfrac{B \qquad [\neg B]^i}{\bot} \\ \Sigma_2 \\ \dfrac{\bot}{B}\, i \\ \Pi_0 \end{array}$$

Case 1) No assumption of Σ_1 is discharged by an application of $(E_\vee)$ that occurs in Σ_2.

Case 2) Some assumption of Σ_1 is discharged by an application r of $(E_\vee)$ that occurs in Σ_2.

Definition 2.5 *Let f be a function which maps formulas of $L(\mathbf{IS4})$ into formulas of $L(\mathbf{I})$ defined as following:*

$$
\begin{aligned}
f(A) &= A, \text{ if } A \text{ is an atomic formula} \\
f(A \rightarrow B) &= f(A) \rightarrow f(B) \\
f(A \wedge B) &= f(A) \wedge f(B) \\
f(A \vee B) &= f(A) \vee f(B) \\
f(\bot) &= \bot \\
f(\Box A) &= f(A) \\
f(\Diamond A) &= f(A)
\end{aligned}
$$

Lemma 2.6 *If $g(\Gamma) \vdash_{IS4} g(A)$, then $f(g(\Gamma)) \vdash_I f(g(A))$ for every $A \in L(\mathbf{I})$*

We will show a more general case:

if $\Gamma \vdash_{IS4} A$, then $f(\Gamma) \vdash_I f(A)$, for every $A \in L(\mathbf{IS4})$.

Let Π be a derivation of A from Γ in **IS4** and r its last inference.

Case 1) r is $(E_{\Diamond})$

$$\Pi \equiv \frac{\begin{array}{c}\Gamma_1\\ \Sigma_1\\ \Box A_1\end{array} \quad \ldots \quad \begin{array}{c}\Gamma_n\\ \Sigma_n\\ \Box A_n\end{array} \quad \begin{array}{c}\Gamma_{n+1}\\ \Sigma_{n+1}\\ \Diamond B\end{array} \quad \begin{array}{c}[\Box A_1]^{i_1}\ldots[\Box A_n]^{i_n}[B]^{i_{n+1}}\\ \Sigma_{n+2}\\ \Diamond C\end{array}}{\Diamond C} r, i_1 \ldots i_{n+1}$$

The derivation Π in **IS4** is transformed into a derivation Π' in **I**.

Without lost of generalization, let n=1

$$\Pi' \equiv \begin{array}{c} \begin{array}{c} f(\Gamma_1)\\ f(\Sigma_1)\\ [f(\Box A_1)] \end{array} \quad \cdots \quad \begin{array}{c} f(\Gamma_2)\\ f(\Sigma_2)\\ \dfrac{f(\Diamond B)}{[f(B)]} def \end{array} \\ f(\Sigma_3)\\ f(\Diamond C) \end{array}$$

Lemma 2.7 $\vdash_I A \leftrightarrow f(g(A))$

Proof by induction on the complexity of A.
Case 1) A is $(B \to C)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[f(g(B))]^i}{B}\,I.H \qquad (B \to C)^j}{C}}{f(g(C))}\,I.H.}{f(g(B)) \to f(g(C))}\,i}{f(g(B) \to g(C))}}{f(\Box(g(B) \to g(C)))}}{f(g(B \to C))}}{(B \to C) \to f(g(B \to C))}\,j$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[f(g(B \to C))]^i}{f(\Box(g(B) \to g(C)))}}{f(g(B) \to g(B))}}{f(g(B)) \to f(g(C))} \qquad \dfrac{[B]^j}{f(g(B))}\,I.H.}{f(g(C))}}{C}\,I.H.}{B \to C}\,j}{f(g(B \to C)) \to (B \to C)}\,i$$

Lemma 2.8 *If* $g(\Gamma) \vdash_{IS4} g(A)$*, then* $\Gamma \vdash_I A$*, for every* $A \in L(I)$

This result follows from:

Lemma 2.6: If $g(\Gamma) \vdash_{IS4} g(A)$, then $f(g(\Gamma)) \vdash_I f(g(A))$, for every $A \in L(\mathbf{I})$

Lemma 2.7: $\vdash_I A \leftrightarrow f(g(A))$

Theorem 2.9 $\Gamma \vdash_I A$ *if and if* $g(\Gamma) \vdash_{NS4} g(A)$

This result follows from:

Lemma 2.2: If $\Gamma \vdash_I A$, then $g(\Gamma) \vdash_{NS4} g(A)$,

Lemma 2.4: If $g(\Gamma) \vdash_{NS4} g(A)$, then $g(\Gamma) \vdash_{IS4} g(A)$, for every $A \in L(I)$.

Lemma 2.8: If $g(\Gamma) \vdash_{IS4} g(A)$, then $\Gamma \vdash_I A$, for every $A \in L(I)$

References

[1] BIERMAN G. M. and de PAIVA V. C. V. On an Ituitionistic Modal Logic. *Studia Logica*, 65: 383-416, 2000.

[2] GÖDEL, K. An interpretation of the intuitionistic proposicional calculus. In - FEFERMAN, S. et al., (eds.)*Collected Works*, vol I. New York: Claredon, 1986(a), p. 301 - 303.

[3] HACKING, I. What is strict implication? *Journal of Symbolic Logic* v. 28, No.1 , pp. 51-71,1963)

[4] KOLMOGOROV, A. N. On the principle of excluded middle. In: HEIJENOORT, J.*From Frege to Gödel: a source book in mathematical logic 1879-1931*. Cambridge: Harvard University, 1977, p 414 - 437.

[5] McKINSEY, J.C.C and TARSKI, A. Some theorems about the sentential calculi of Lewis and Heyting *The Journal of Symbolic Logic* v. 13, 1948, p 1-15.

[6] MEDEIROS, M. P. N. A New S4 Classical Modal Logic in Natural Deduction. *Journal of Symbolic Logic*, v. 71, n.3, 2006.

[7] PRAWITZ, D. *Natural Deduction*, Almqvist & Wiksel, Stockholm, 1965

[8] PRAWITZ, D. and MALMNÄS, P. E. A survey of some connections between classical, intuitionistic and minimal logic. In: SCHNDT, H. *Contributions to mathematical logic*. Amsterdam: North-Holland, 1968, p. 215-299.