Determinacy of Wadge classes in Baire space and simple iteration of inductive definition

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August 4, 2009

Abstract

Determinacy of Wadge classes in the Cantor space below $\Sigma^0_2 \wedge \Pi^0_2$ make a nice hierarchy.
In this talk, we consider determinacy of those classes in the Baire space.

1. Outline of this talk

- Determinacy?
- Second order arithmetic?
- Wadge classes?
- Determinacy of Wadge classes in the Cantor space
- Determinacy of Wadge classes in the Baire space
  - $\mathsf{Sep}(\Delta^0_1, \Sigma^0_2)$-Det $\Leftrightarrow$ $\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$-Det $\Leftrightarrow$ Σ^0_2-Det
  - $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-Det and Σ^1_1-TID

2. Infinite games?

Let X be either $\mathbb{N}$ or $\{0, 1\}$.
For a given formula $\varphi(f)$,

- Players I and II alternately choose $x \in X$ to form $f \in X^{\mathbb{N}}$.

I	$f(0)$		$f(2)$		$f(4)$		$\cdots$	
II		$f(1)$		$f(3)$		$f(5)$		$\cdots$

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- I wins iff $\varphi(f)$.
- If one of the players has a winning strategy in the above game, $\varphi(f)$ is *determinate*.

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Γ *determinacy* asserts that every $\varphi(f) \in \Gamma$ is determinate.

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Basic arithmetic

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Addition $n + 0 = n, \quad n + (m + 1) = (n + m) + 1$,

Multiplication $n \cdot 0 = 0, \quad n \cdot (m + 1) = n \cdot m + n$,

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Δ^0_1 comprehension

$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X(\varphi(n) \leftrightarrow n \in X)$,
for $\varphi \in \Sigma^0_1$ and $\psi \in \Pi^0_1$.

5. Table of determinacy

We have the following equivalences over RCA_0:

System	Det. in $2^{\mathbb{N}}$ (-Det*)	Det. in $\mathbb{N}^{\mathbb{N}}$ (-Det)
ATR_0	Δ^0_2, Σ^0_2	Δ^0_1, Σ^0_1
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Π^1_1-TR_0	$\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$	Δ^0_2
Σ^1_1-ID	$\Sigma^0_2 \wedge \Pi^0_2$	Σ^0_2
$[\Sigma^1_1]^2$-ID	$(\Sigma^0_2 \wedge \Pi^0_2) \vee \Sigma^0_2$	$\Sigma^0_2 \wedge \Pi^0_2$

6. Subsystems of second order arithmetic

ATR$_0$ RCA$_0+$ arithmetical transfinite recursion:

"For any arithmetical operator $\Psi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ and a well-ordering X, we can iterate Ψ along X."

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Σ^1_1-ID$_0$ RCA$_0$ + Σ^1_1 inductive definition:

“For any Σ^1_1 operator $\Psi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, we have a fixed point, i.e., W s.t. $\Psi(W) = W$”

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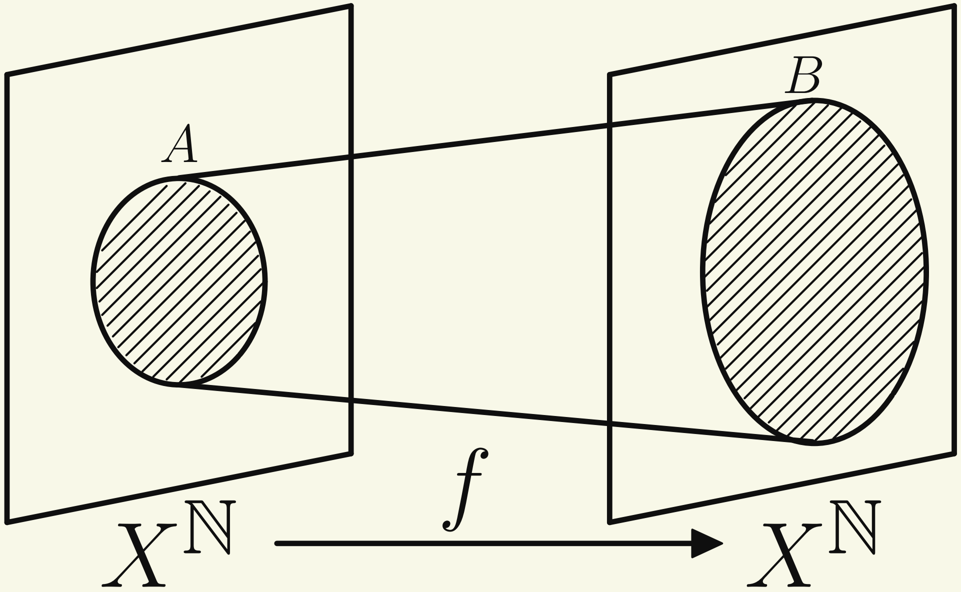
Π^1_1-TR_0	$\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$	Δ^0_2
Σ^1_1-ID	$\Sigma^0_2 \wedge \Pi^0_2$	Σ^0_2
$[\Sigma^1_1]^2$-ID	$(\Sigma^0_2 \wedge \Pi^0_2) \vee \Sigma^0_2$	$\Sigma^0_2 \wedge \Pi^0_2$

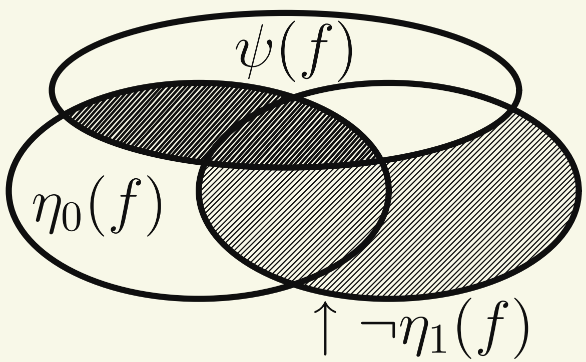
7. Wadge reducibility?

For given $A, B \subseteq X^{\mathbb{N}}$, which is "simpler?"
We say *A is Wadge reducible to B* if there is a continuous function $f : X^{\mathbb{N}} \to X^{\mathbb{N}}$ with $f^{-1}(B) = A$.



$\mathsf{Sep}(\Gamma, \Gamma')$

$(\psi(f) \wedge \eta_0(f)) \vee (\neg\psi(f) \wedge \neg\eta_1(f))$



$\psi \in \Gamma$ and $\eta_i \in \Gamma'$

$$\Sigma^0_2 \wedge \Pi^0_2$$

$$\vdots$$

$$\vdots$$

$$\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$$

$$\vdots$$

$$\mathsf{Sep}(\Delta^0_1, \Sigma^0_2)$$

$$\Sigma^0_2$$

Δ^0_2 $\left\{ \begin{array}{c} \vdots \\ (\Sigma^0_1 \wedge \Pi^0_1) \\ \mathsf{Sep}(\Delta^0_1, \Sigma^0_1) \\ \Sigma^0_1 \\ \Delta^0_1 \end{array} \right.$

(all of the above) $\Big\}$ $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$

Remark: $\mathsf{Sep}(\Delta^0_n, \Sigma^0_n) = \neg(\Sigma^0_n \wedge \Pi^0_n) \cap (\Sigma^0_n \wedge \Pi^0_n)$.

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Theorem

Sep(Δ^0_1, Σ^0_2)-Det $\Leftrightarrow$ Sep(Σ^0_1, Σ^0_2)-Det $\Leftrightarrow$ Σ^0_2-Det $\Leftrightarrow$ Σ^1_1-ID$_0$

Proof is similar to the case of the Cantor space.

Key point of Proof

Actually, for any Σ^0_2 game $\varphi(f)$, Σ^1_1-ID$_0$ proves the existence of the *winning set* W for player I, i.e.,

- $s \in W$ →Player I wins $\varphi(f)$ at s
- $s \notin W$ →Player II wins $\varphi(f)$ at s

By the above W, we can reduce Sep(Δ^0_1, Σ^0_2) and Sep(Σ^0_1, Σ^0_2) games to Δ^0_1 and $\Sigma^0_1 \wedge \Pi^0_1$ game, respectively.

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Π^1_1-CA_0	$\mathsf{Sep}(\Sigma^0_1, \Sigma^0_2)$	$\mathsf{Sep}(\Delta^0_1, \Sigma^0_1)$, $\Sigma^0_1 \wedge \Pi^0_1$
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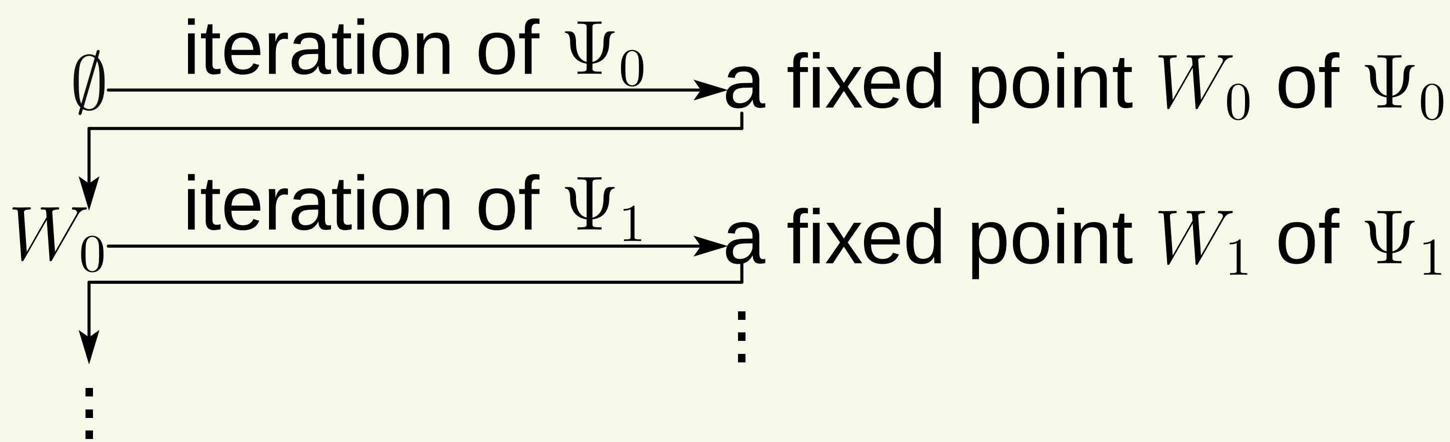
How about $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-Det?

11. Σ^1_1-TID

Σ^0_1 transfinite inductive definition (Σ^0_1-TID)

“For any well ordering X and sequence $\langle \Psi_x : x \in X \rangle$, we have the sequence of set $\langle W_x : x \in X \rangle$ s.t.

$$W_x = \text{the fixed point of } \Psi_x \text{ starting with } \bigcup_{y<x} W_y$$



12. Σ^1_1-TID and $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-Det

Theorem $\mathsf{RCA}_0 + \Sigma^1_1$-TID proves $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$-Det

Idea for the proof

- Δ^0_2 formula can be expressed as a disjoint union of transfinitely many $\Sigma^0_1 \wedge \Pi^0_1$ formulas.
- Therefore, $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$ game $(\psi(f) \wedge \eta_0(f)) \vee (\neg\psi(f) \wedge \eta_1(f))$ can be expressed as a disjoint union of transfinitely many Σ^0_2 and Π^0_2 formulas.
- Since, for any Σ^0_2 game, Σ^1_1-ID yields the winning set for playse I, iteration of Σ^1_1 inductive definition yields the winning set for player I in a $\mathsf{Sep}(\Delta^0_2, \Sigma^0_2)$ game.

13. Strength of Σ^1_1-TID

By modifying the proof of Σ^0_2-Det $\rightarrow$ Σ^1_1-ID, we may have the proof of the following conjecture:

Conjecture Σ^1_1-TID is equivalent to Σ^1_1-ID over RCA_0.

If the above conjecture is true, there is no hierarchy of determinacy between Σ^0_2 and $\Sigma^0_2 \wedge \Pi^0_2$ game in the Baire space, contrary to the case of the Cantor space.

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