On consistency of Peano's Arithmetic System

Teodor J. Stępień [1] Łukasz T. Stępień [2]

[1]Pedagogical University, Kraków, Poland,
tstepien@ap.krakow.pl,
[2]Pedagogical University, Kraków, Poland,
lstepien@ap.krakow.pl, stepien50@poczta.onet.pl

05.08.2009 / Logic Colloquium 2009, Sofia, Bulgaria

Outline

Abstract

In this talk we establish that Peano's Arithmetic System is consistent in the traditional sense and that Peano's Arithmetic System is structurally incomplete.

Bibliography I

- W. A. Pogorzelski and T. Prucnal,
  "Structural completeness of first-order predicate calculus"
  Z. Math. Logik Grundl. Math., **21** (1975).
- T. Stępień,
  "Derivability"
  Rep. Math. Logic, **33**, 79-98 (1999).
- J. Von Neumann,
  "Die formalistische Grundlegung der Mathematik",
  Erkenntnis, **2**, 116-121 (1931).
- Yu. L. Ershov and E. A. Palyutin,
  "Mathematical Logic"
  (in English)
  Mir Publishers, Moscow 1984

Bibliography II

- R. Murawski,
  "On the incompleteness of Arithmetic once more"
  in *Essays on philosophy and logic*
  Proceedings of the XXXth Conference on the history of logic, dedicated to Roman Suszko, Cracow, October 19-21, 1984
  Edited by J. Perzanowski
  Jagiellonian University Press, Cracow 1987
- W. A. Pogorzelski,
  "The classical calculus of quantifiers"
  (in Polish)
  PWN, Warszawa 1981
- H. Rasiowa,
  "Introduction to contemporary mathematics"
  (in Polish)
  PWN, Warszawa 1977.

Terminology I

Let: $\rightarrow, \sim, \vee, \wedge, \equiv$ denote connectives implication, negation, disjunction, conjuction and equivalence, respectively.
Symbols $x_1, x_2, ...$ are individual variables. Symbols $a_1, a_2, ...$ are individual constants. Symbols $f_k^n (n, k \in N)$ are n-ary functional letters. Symbols $t_1, t_2, ...$ are terms. Symbols P_k^n $(n, k \in N = \{1, 2, 3, ...\})$ are n-ary predicate letters. The set of all atomic formulas of the form $P_k^n(t_1, ..., t_n)$ is denoted by At_1. The symbols $\bigwedge, \bigvee$ are quantifiers. The set S_1 of all well-formed formulas is constructed in the usual manner from the symbols listed above. Next $vf(\phi)$ denotes the set of all free variables occuring in ϕ.
By $At_0 = \{p_1^1, p_2^1, ..., p_1^2, p_2^2, ..., p_1^k, p_2^k, ...\}$ we denote the set of all propositional variables. Hence S_0 is the set of all well-formed formulas that are built in the usual manner from propositional variables by means of logical connectives. Next R_{S_1} denotes the set of all rules over S_1. For any $X \subseteq S_1$, $Cn(R, X)$ is the smallest subset of S_1 containing X and closed under the rules of $R \subseteq R_{S_1}$. Next the couple $\langle R, X \rangle$ is called a system, whenever $X \subseteq S_1$ and $R \subseteq R_{S_1}$. By $r_\star$ we denote the rule of simultaneous substitution for predicate letters. Namely $\langle \{\alpha\}, \beta \rangle \in r_\star \Leftrightarrow \beta = h^e(\alpha)$ for some endomorphism $h^e : S_1 \longrightarrow S_1$ which is an extension of the function $e : At_1 \longrightarrow S_1, e \in \mathcal{E}_\star$ (for details see [Pogorzelski 1981]). Next here and later r_0 denotes Modus Ponens and r_+ denotes the generalization rule.

Terminology II

$\{r_0, r_+\} = R_{0+}$. Here and later we use $\Rightarrow, \neg, \Leftrightarrow, \&, \mathcal{V}, \forall, \exists$ as metalogical symbols and $A \neq B$ denotes that $A \nsubseteq B$ or $B \nsubseteq A$. We write $X \subset Y$ for $X \subseteq Y$ and $Y \neq X$.
We define function $i : S_1 \longrightarrow S_0$ as follows:

(a) $i(P^n_k(t_1, ..., t_n)) = p^n_k, (p^n_k \in At_0)$,

(b) $i(\sim \phi) = \sim i(\phi)$,

(c) $i(F\phi\psi) = Fi(\phi)i(\psi), F \in \{\rightarrow, \wedge, \vee, \equiv\}$,

(d) $i(\bigwedge t_k \phi) = i(\bigvee t_k \phi) = i(\phi)$.

Terminology III

Let Z_2 denotes the set of all formulas valid in the classical propositional calculus and let L_2 denotes the set of all formulas valid in the classical functional calculus. Next $Z_2^\star = \{\phi \in S_1 : i(\phi) \in Z_2\}$ and $\bar{S}_1 = \{\phi \in S_1 : vf(\phi) = \emptyset\}$ and $\bar{Z}_2 = Z_2^\star \cap \bar{S}_1$, (for the details see [Pogorzelski 1981].

Now we repeat some well known properties of operation of consequence and some well-known definitions (see [Pogorzelski 1981], [Pogorzelski and Prucnal, 1975]). Let $R \subseteq R_{S_1}$ and $X \subseteq S_1$. Then:

(a_1) $X \subseteq Cn(R, X)$

(a_2) $X \subseteq Y \Rightarrow Cn(R, X) \subseteq Cn(R, Y)$

(a_3) $R \subseteq R' \Rightarrow Cn(R, X) \subseteq Cn(R', X)$

(a_4) $Cn(R, Cn(R, X)) \subseteq Cn(R, X)$

(a_5) $Cn(R, X) = \cup\{Cn(R, Y) : Y \in Fin(X)\}$

Terminology IV

where $Y \in Fin(X)$ denotes that Y is the finite subset of X.

Def. 1 $\langle R, X\rangle \in Cns^T \Leftrightarrow (\neg\exists\alpha \in S_A)[\alpha \in Cn(R, X)\ \&\ \sim\alpha \in Cn(R, X)]$

Def. 2 $\langle R, X\rangle \in Cns^A \Leftrightarrow Cn(R, X) \neq S_A$

Def. 3 $\langle R, X\rangle \in Cpl^T \Leftrightarrow (\forall\alpha \in \bar{S}_1)[\alpha \in Cn(R, X)\ \mathcal{V}\ \sim\alpha \in Cn(R, X)]$

Def. 4 $\langle R, X\rangle \in Cpl^A \Leftrightarrow (\forall\alpha \in S_1 - Cn(R, X))Cn(R, X \cup \{\sim\alpha\}) = S_1$

Def. 5

$$r \in Perm(R, X) \quad \text{iff} \tag{1}$$

$$(\forall\pi \subseteq S_1)(\forall\phi \in S_1)[\langle\pi, \phi\rangle \in r\ \&\ \pi \subseteq Cn(R, X) \Rightarrow \phi \in Cn(R, X) \tag{2}$$

Def. 6

$$r \in Der(R, X) \quad \text{iff} \tag{3}$$

$$(\forall\pi \subseteq S_1)(\forall\phi \in S_1)[\langle\pi, \phi\rangle \in r \Rightarrow \phi \in Cn(R, X \cup \pi)] \tag{4}$$

Terminology V

Def. 7

$$r \in Struct_{S_1} \quad \text{iff} \tag{5}$$

$$(\forall \pi \subseteq S_1)(\forall \phi \in S_1)(\forall e \in \mathcal{E}_\star)[\langle \pi, \phi \rangle \in r \Rightarrow \langle h^e(\pi), h^e(\phi) \rangle \in r] \tag{6}$$

Def. 8

$$\langle R, X \rangle \in SCpl \quad \text{iff} \quad Struct_{S_1} \cap Perm(R, X) \subseteq Der(R, X) \tag{7}$$

Basic Theorems I

Now we repeat some well-known theorems (see [Pogorzelski 1981]).
THEOREM 1 $\langle R_{0+}, L_2 \rangle \in Cns^T$

THEOREM 2 $\langle R_{0+}, L_2 \rangle \in Cns^A$

THEOREM 3:

$$(\forall \alpha \in \bar{S}_A)(\forall \beta \in S_A)(\forall X \subseteq S_A)[\beta \in Cn(R_{0+}, L_2 \cup X \cup \{\alpha\}) \Rightarrow$$

$$(\alpha \rightarrow \beta \in Cn(R_{0+}, L_2 \cup X))]$$

THEOREM 4

$$(\forall \alpha \in \bar{S}_A)(\forall X \subseteq S_A)[Cn(R_{0+}, L_2 \cup X \cup \{\alpha\}) = S_A \Leftrightarrow \sim \alpha \in Cn(R_{0+}, L_2 \cup X)]$$

THEOREM 5

$$(\forall \alpha \in \bar{S}_A)(\forall X \subseteq S_A)[\alpha \notin Cn(R_{0+}, L_2 \cup X) \Leftrightarrow Cn(R_{0+}, L_2 \cup X \cup \{\sim \alpha\}) \neq S_A]$$

Arithmetic terminology I

Next S_A denotes the set of all well-formed formulas of Peano's Arithmetic System. Hence, $\bar{S}_A = \{\phi \in S_A : vf(\phi) = \emptyset\}$.
Analogically, R_{S_A} denotes the set of all rules over S_A. For any $X \subseteq S_A$ and for any $R \subseteq R_{S_A}$, $Cn(R, X)$ is the smallest subset of S_A containing X and closed under the rules of R. The couple $\langle R, X \rangle$ is called a system, whenever $R \subseteq R_{S_A}$ and $X \subseteq S_A$. $R_{0+} = \{r_0, r_+\} \subseteq R_{S_A}$. Next:

(ψ_1) $\bigwedge x(x + 0 = x)$,

(ψ_2) $\bigwedge x \bigwedge y(x \cdot Sy = x \cdot y + x)$,

(ψ_3) $\bigwedge x \bigwedge y(Sx = Sy \rightarrow x = y)$,

(ψ_4) $\bigwedge x \bigvee y(y = Sx)$

(ψ_5) $\bigwedge x \bigwedge y[x + Sy = S(x + y)]$

(ψ_6) $\bigwedge x(x \cdot 0 = 0)$

(ψ_7) $\sim \bigvee x(Sx + 1 = 1)$

(ψ_8) $\bigwedge x_1 \bigwedge x_2[\bigvee x_3(Sx_3 + x_1 = x_2) \equiv (x_1 < x_2)]$

(ψ_9) $\bigwedge x \sim (Sx = 0)$

Arithmetic terminology II

$(\psi_{10})\ \ \phi(0)\&\bigwedge x(\phi(x)\Rightarrow\phi(Sx))\Rightarrow\bigwedge x\phi(x)$

$$X_P = \{\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8,\psi_9\}$$

A_{14} denotes here the set of all axioms of induction.

Arithmetic terminology III

At last L_r and A_r denote the set of all logical axioms and the set of all specific axioms in Peano's Arithmetic System, respectively. Hence $\langle R_{0+}, L_r \cup A_r \rangle$ is the Peano's Arithmetic System, where $A_r = X_P \cup A_{14}$ and $Cn(R_{0+}, L_r \cup A_r) = A_r^{\star}$, (see [Rasiowa 1977], [Ershov and Palyutin, 1984], [Murawski, 1987]).
Next:

(1_1) $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$

(1_2) $(\sim \alpha \to \alpha) \to \alpha$

(1_3) $\alpha \to (\sim \alpha \to \beta)$

(1_4) $\alpha \wedge \beta \to \alpha$

(1_5) $\alpha \wedge \beta \to \beta$

(1_6) $\alpha \to (\beta \to \alpha \wedge \beta)$

(1_7) $\bigwedge x_k \phi \to \phi(\frac{x_k}{t_n})$, if $x_k \in Ff(t_n, \phi)$

(1_8) $\bigwedge x_k (\psi \to \psi) \to (\phi \to \bigwedge x_k \psi)$, if $x_k \notin vf(\phi)$

Arithmetic terminology IV

where:

$$\alpha, \beta, \gamma, \phi, \psi \in S_A,$$

$x_k \in Ff(t_n, \phi)$ denotes that x_k is free for the terms t_n in the formula ϕ.
Hence:

$$L_1^2 = \{1_1, 1_2, 1_3, 1_4, 1_5, 1_6, 1_7, 1_8\}$$

THE MAIN RESULT I

Theorem (I)

$$\langle R_{0+}, L_r \cup A_r \rangle \in Cns^T$$

Proof. Let:

(1) $\langle R_{0+}, L_r \cup A_r \rangle \notin Cns^T$

Hence

(2) $Cn(R_{0+}, L_r \cup A_r) = S_A$
Now we introduce many formulas, which are used in the proof. Thus:

(3) $\overset{\square}{u}_{27} = \sim (1 < 1)$

(4) $\overset{\square}{\psi}_8 = \overset{\square}{u}_{27} \to \psi_8$

(5) $\gamma_2' = (\psi_7 \equiv \sim\sim \psi_1) \to \overset{\square}{u}_{27}$

(6) $\gamma_0' = (\psi_7 \to \psi_1) \to \psi_8$

THE MAIN RESULT II

(7) $\gamma_0 = \overset{\square}{u}_{27} \rightarrow \gamma'_0$

(8) $\gamma'_4 = \gamma'_0 \rightarrow [\gamma'_2 \rightarrow (\psi_7 \equiv\sim\sim \psi_1)]$

(9) $\gamma_4 = \gamma'_2 \rightarrow [\gamma_0 \rightarrow (\psi_7 \equiv\sim\sim \psi_1)]$

(10) $w = \sim [\sim \overset{\square}{u}_{27} \equiv\sim \psi_7] \rightarrow\sim [\overset{\square}{u}_{27} \equiv \psi_7]$

(11) $\psi_1^v = \gamma'_2 \rightarrow (\sim \gamma_4 \rightarrow (\gamma'_4 \rightarrow (\psi_1 \rightarrow (\sim \psi_7 \rightarrow ((\overset{\square}{u}_{27} \equiv \psi_7) \rightarrow ((\psi_7 \equiv\sim \psi_1) \rightarrow (\sim \overset{\square}{u}_{27} \rightarrow (\sim (\psi_7 \equiv\sim\sim \psi_1) \rightarrow (\overset{\square}{\psi}_8 \rightarrow \gamma'_0)))))))))$

(12) $\phi_0^\star = \psi_1 \rightarrow (\sim \psi_7 \rightarrow (\gamma'_2 \rightarrow ((\psi_7 \equiv\sim \psi_1) \rightarrow (\sim \gamma_4 \rightarrow (\sim \overset{\square}{u}_{27} \rightarrow (\overset{\square}{\psi}_8 \rightarrow ((\overset{\square}{u}_{27} \equiv \psi_7) \rightarrow (\sim (\psi_7 \equiv\sim\sim \psi_1) \rightarrow w))))))))$

(14) $O_1 =$

-
-
-

THE MAIN RESULT III

(26) $O_{14} = \gamma'_2 \to (\gamma'_4 \to (\sim \gamma'_0 \to (\sim (\psi_7 \equiv \sim\sim \psi_1) \to (\psi_1 \to (\sim \psi_7 \to (\sim \gamma_4 \to (\overset{\square}{\psi}_8 \to (w \to ((\psi_7 \equiv \sim \psi_1) \to (\sim (\overset{\square}{u}_{27} \equiv \psi_7) \to \overset{\square}{u}_{27}))))))))))$

-
-
-

(30) $O_{18} = (\psi_1 \to \psi_7) \to ((\psi_7 \to \psi_1) \to (\psi_7 \equiv \sim\sim \psi_1))$

(31) $O_{19} =$

(32) $O_{20} = \gamma'_2 \to (\sim \gamma_4 \to (\gamma'_4 \to (\sim \gamma'_0 \to (\psi_1 \to (\sim \psi_7 \to (\sim \overset{\square}{u}_{27} \to (\overset{\square}{\psi}_8 \to ((\overset{\square}{u}_{27} \equiv \psi_7) \to (\sim (\psi_7 \equiv \sim\sim \psi_1) \to (\sim w \to (\psi_7 \equiv \sim \psi_1)))))))))))$

-
-
-

(35) $O_{33} = \gamma'_2 \to (w \to (\psi_1 \to (\sim \psi_7 \to (\sim \overset{\square}{u}_{27} \to (\overset{\square}{\psi}_8 \to (\sim \gamma'_0 \to (\gamma'_4 \to (\sim \gamma_4 \to (\sim (\psi_7 \equiv \sim\sim \psi_1) \to (\psi_7 \equiv \sim \psi_1))))))))))$

Lemma1 I

Now we formulate the first Lemma:

LEMMA1. $\gamma_4' \to \gamma_4 \in$
$Cn(R_{0+}, L_1^2 \cup \{\phi_0^\star \to (O_{14} \to (\to O_{33} \to (\sim \gamma_0' \to (\sim \overset{\Box}{u}_{27} \to (\sim \gamma_4 \to O_{18}))))), \phi_0^\star \to (O_{20} \to O_{14}), \phi_0^\star \to (O_{20} \to O_{33}), \phi_0^\star \to O_{20}, O_{14} \to (O_{20} \to (O_{33} \to \psi_1^v)) \ \})$

Using LEMMA 1 and many other Lemmas, one can prove the following (k) step.

(k) $(\exists L' \subseteq L_r)[L_1^2 \subseteq L' \ \& \ Cn(R_{0+}, L' \cup \{\overset{\Box}{u}_{27}\}) = S_A]$

Hence, by THEOREM 4:

$(k+1)$ $1 < 1 \in Cn(R_{0+}, L'),$

what is impossible.

$\Box$

ON STRUCTURAL INCOMPLETENESS OF PEANO'S ARITHMETIC SYSTEM I

In [Stępień, 1999] it was proved the following:

Theorem(II):

Let $X \subseteq S_1$ and $Cn(R_{0+}, L_2 \cup X) = Z_3$.

Then:

$$\langle R_{0+}, L_2 \cup X \rangle \in SCpl \quad \text{iff} \quad (\forall \alpha \in \bar{Z}_2)[\alpha \in Z_3 \quad \mathcal{V} \quad \sim \alpha \in Z_3]$$

Thus:

Theorem (III)

$\langle R_{0+}, L_r \cup A_r \rangle \notin SCpl$

Proof.

By First Gödel THEOREM, Theorem(I) and Theorem(II). □

Summary

Peano's Arithmetic System is consistent

and

Peano's Arithmetic System is structurally incomplete.