A Logic Based
Foundation and Analysis of

RELATIVITY THEORY

by Hajnal Andréka, István & Péter Németi

WELCOME

- Our Research Group:

  H. Andréka, J. X. Madarász, I. & P. Németi,

  G. Székely, R. Tordai.

  Cooperation with:

  L. E. Szabó, M. Rédei.

AIMS OF OUR SCHOOL

- Analysis of the logical structure of R. Theory
- Base the theory on simple, unambiguous axioms with clear meanings
- Make Relativity Theory:
  - More transparent logically
  - Easier to understand and teach
  - Easier to change
  - Modular
- Demystify Relativity Theory

PLAN OF OUR PRESENTATIONS

Logical analysis of:

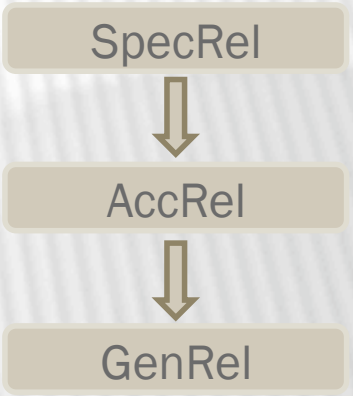
- Special relativity theory

  Transition to:

- Accelerated observers and Einstein's EP

  Transition to:

- General relativity
- Exotic space-times, black holes, wormholes
- Application of general relativity to logic
- Visualizations of Relativistic Effects
- Relativistic dynamics, Einstein's $E=mc^2$



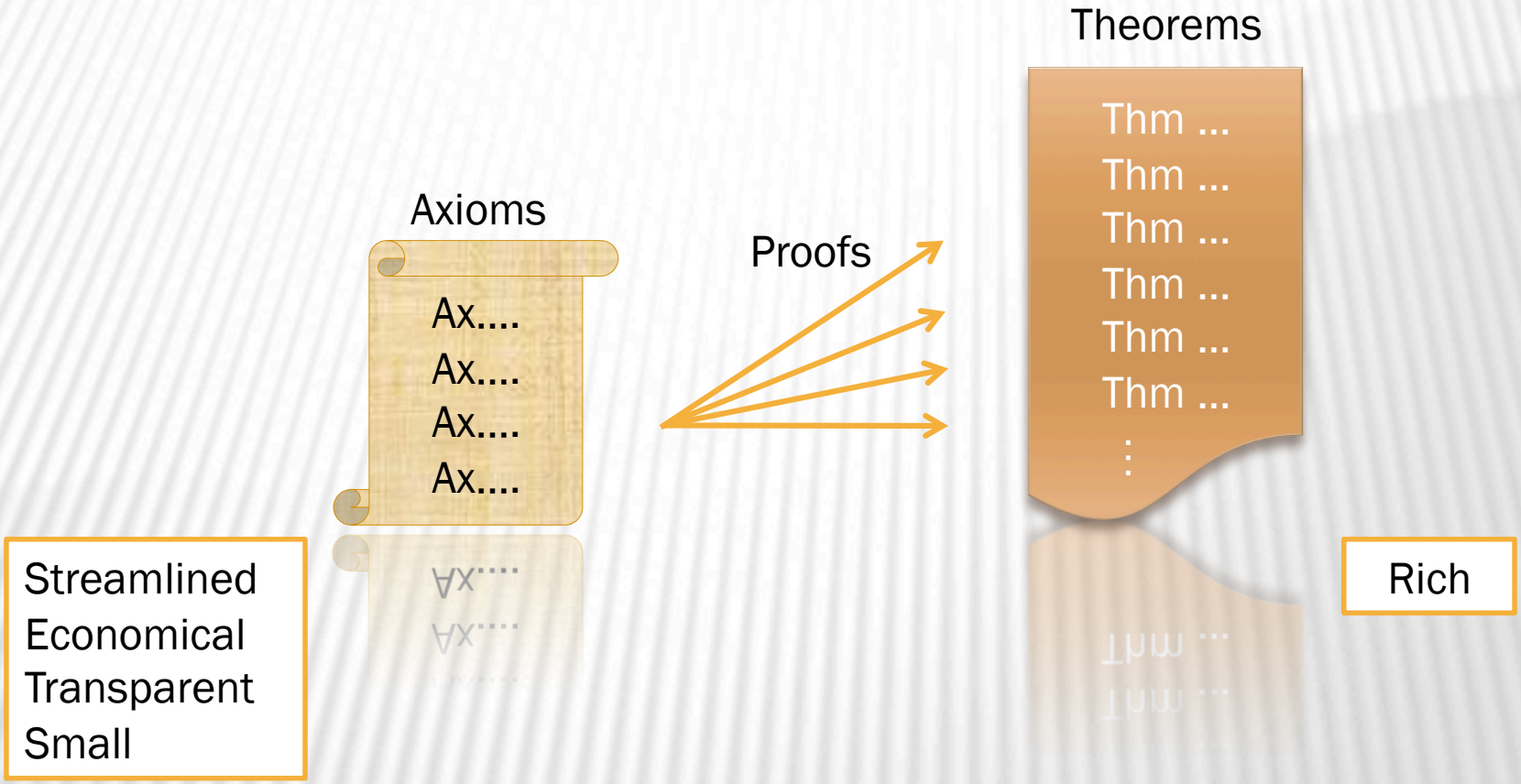
AIMS OF OUR SCHOOL

- R.T.'s as theories of First Order Logic

S. R. → $SpecRel \subseteq FOL$

G. R. → $GenRel \subseteq FOL$

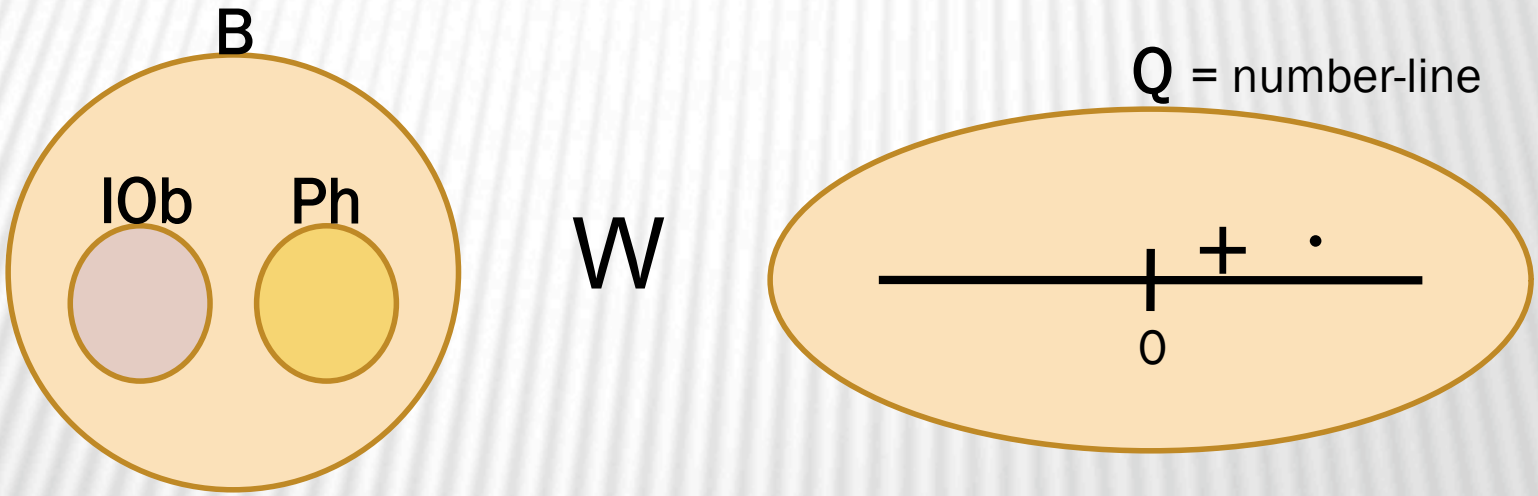
LOGIC AXIOMATIZATION OF R.T.



LANGUAGE FOR SPECREL

$$\langle B, IOb, Ph, Q, +, \cdot, W \rangle$$

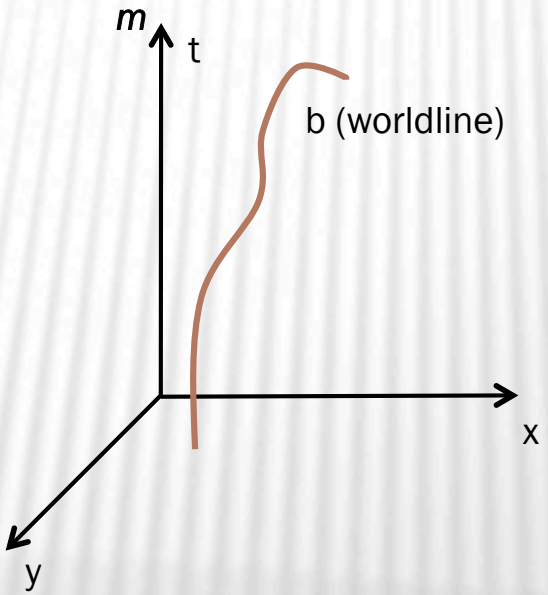
Bodies (test particles), Inertial Observers, Photons, Quantities, usual operations on it, Worldview



LANGUAGE FOR SPECREL

$$W \subseteq IOb \times Q^4 \times B$$

W(m, t x y z, b) ⇔ body “b” is present at coordinates “t x y z” for observer “m”



$$worldline: \quad wline_m(b) := \{p \in Q^4 : W(m,p,b)\}$$

AXIOMS FOR SPECREL

- AxField

Usual properties of addition and multiplication on Q :
Q is an ordered Euclidean field.

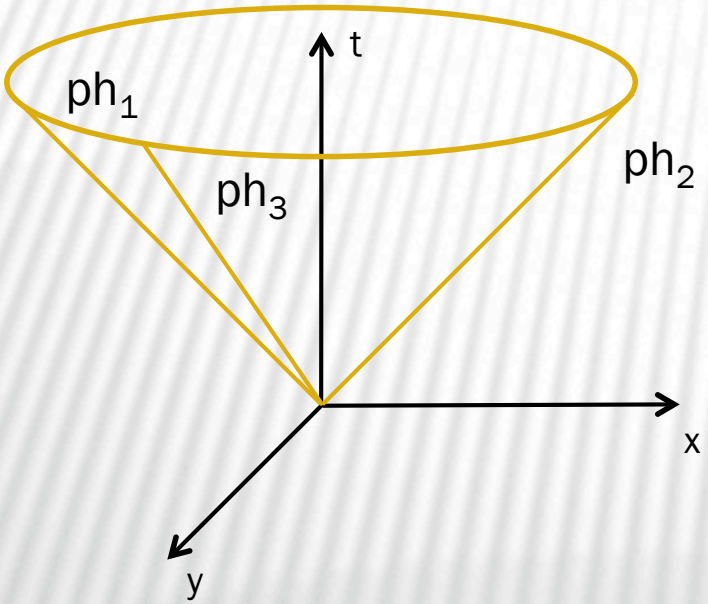
1. (Q , + , ·) is a field in the sense of abstract algebra (with 0 , + , 1 , / as derived operations)
2. $0 = x^2 + y^2 + z^2 \longrightarrow x = y = z = 0$
3. $\exists y(x = y^2 \; or \; -x = y^2)$

Ordering derived: $x \leq y \overset{d}{\Leftrightarrow} \exists z(y - x = z^2)$

AXIOMS FOR SPECREL

- AxPh

For all inertial observers the speed of light is the same in all directions and is finite.
In any direction it is possible to send out a photon.



Formalization:

$$(\forall m \in IOb)(\exists c \in Q)(\forall p, q \in Q^4)$$

$$\left[\begin{array}{c} |p_s - q_s| = c \cdot |p_t - q_t| \\ \leftrightarrow \\ (\exists ph \in Ph)\ p, q \in wline_m(ph) \end{array}\right]$$

$$where\ p_s = \langle p_2, p_3, p_4 \rangle\ and\ p_t = p_1.$$

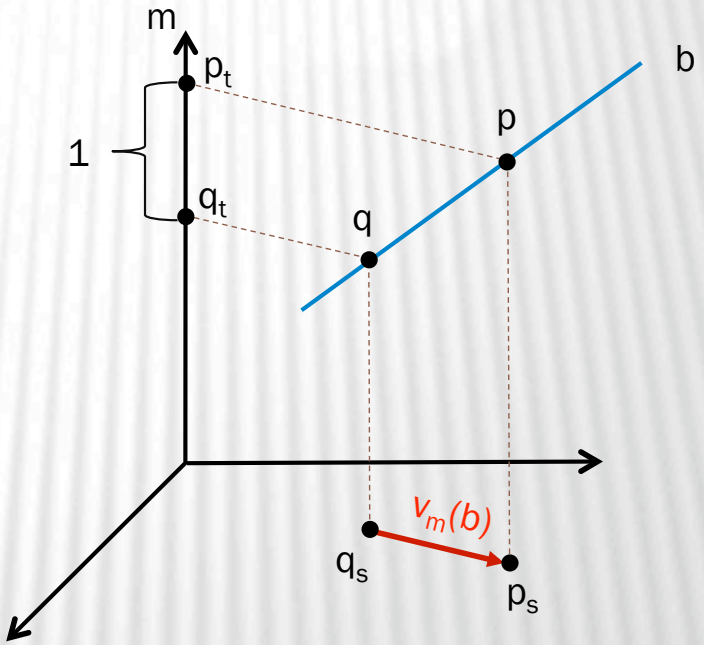
AXIOMS FOR SPECREL

What is speed?

$$p = \langle p_1, p_2, p_3, p_4 \rangle$$

$$p_t = p_1$$

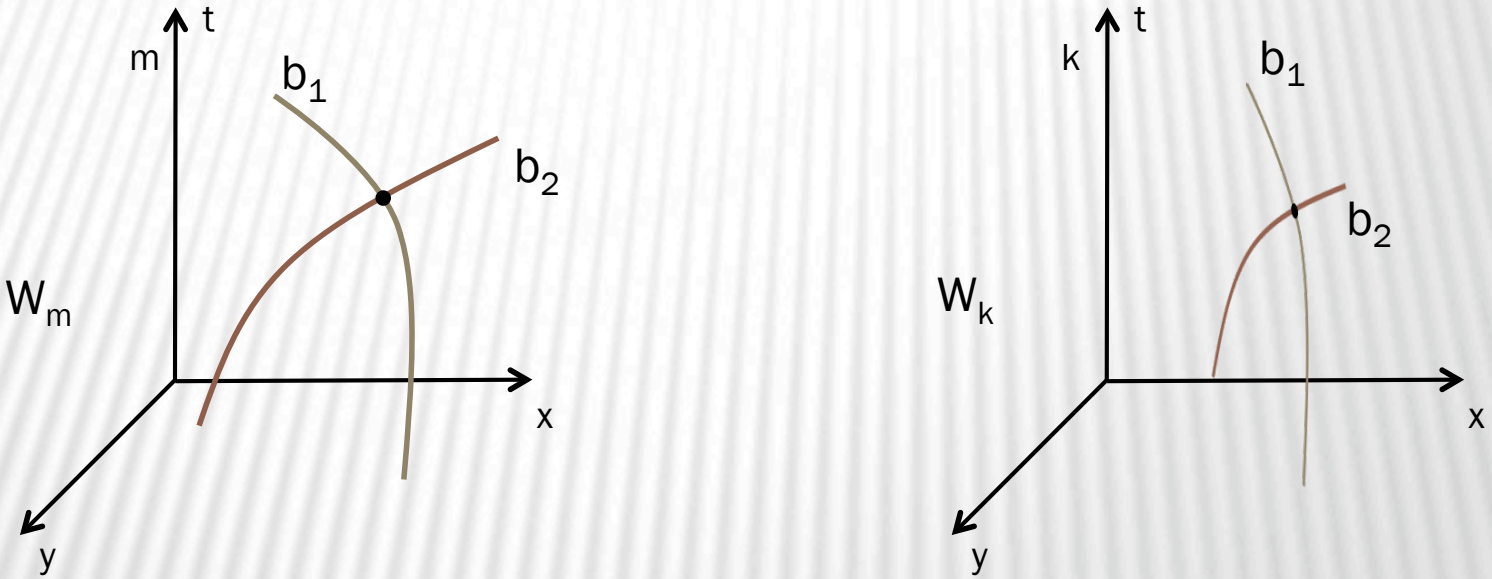
$$p_s = \langle p_2, p_3, p_4 \rangle$$



AXIOMS FOR SPECREL

- AxEv

All inertial observers coordinatize the same events.



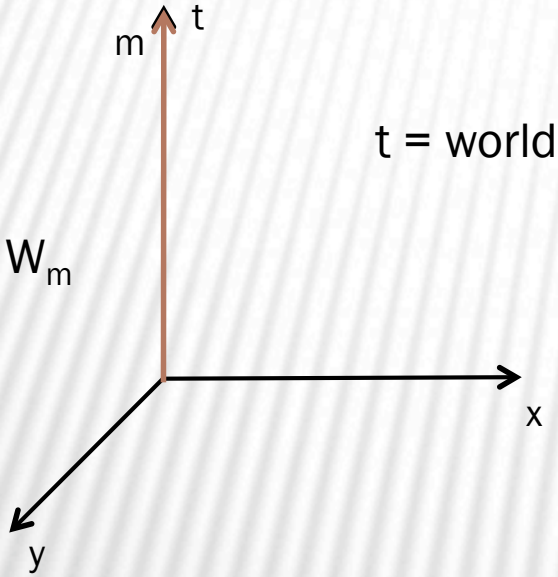
Formalization: $(\forall m,k \in IOb)(\forall txyz)$

$$\exists t'x'y'z' \; \forall b[W(m,t\,x\,y\,z,b) \leftrightarrow W(k,t'x'y'z',b)]$$

AXIOMS FOR SPECREL

- AxSelf

An inertial observer sees himself as standing still at the origin.



t = worldline of the observer

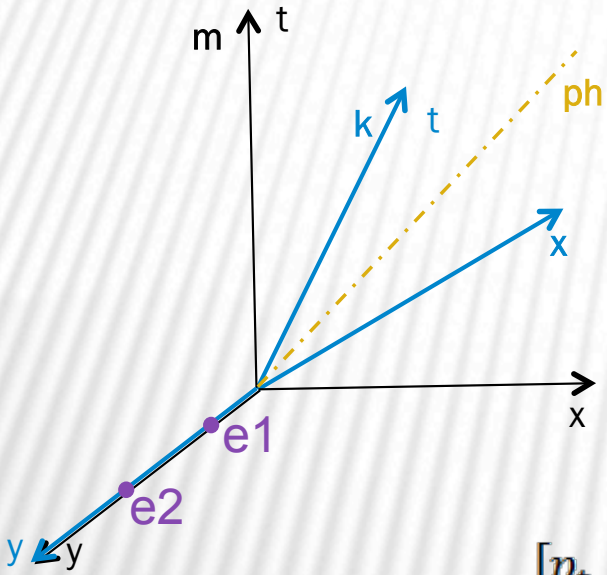
Formalization: $(\forall m \in IOb)(\forall txyz)$

$$W(m, t\,x\,y\,z, m) \leftrightarrow x = y = z = 0$$

AXIOMS FOR SPECREL

- AxSymd

Any two observers agree on the spatial distance between two events, if these two events are simultaneous for both of them, and $|v_m(ph)|=1$.



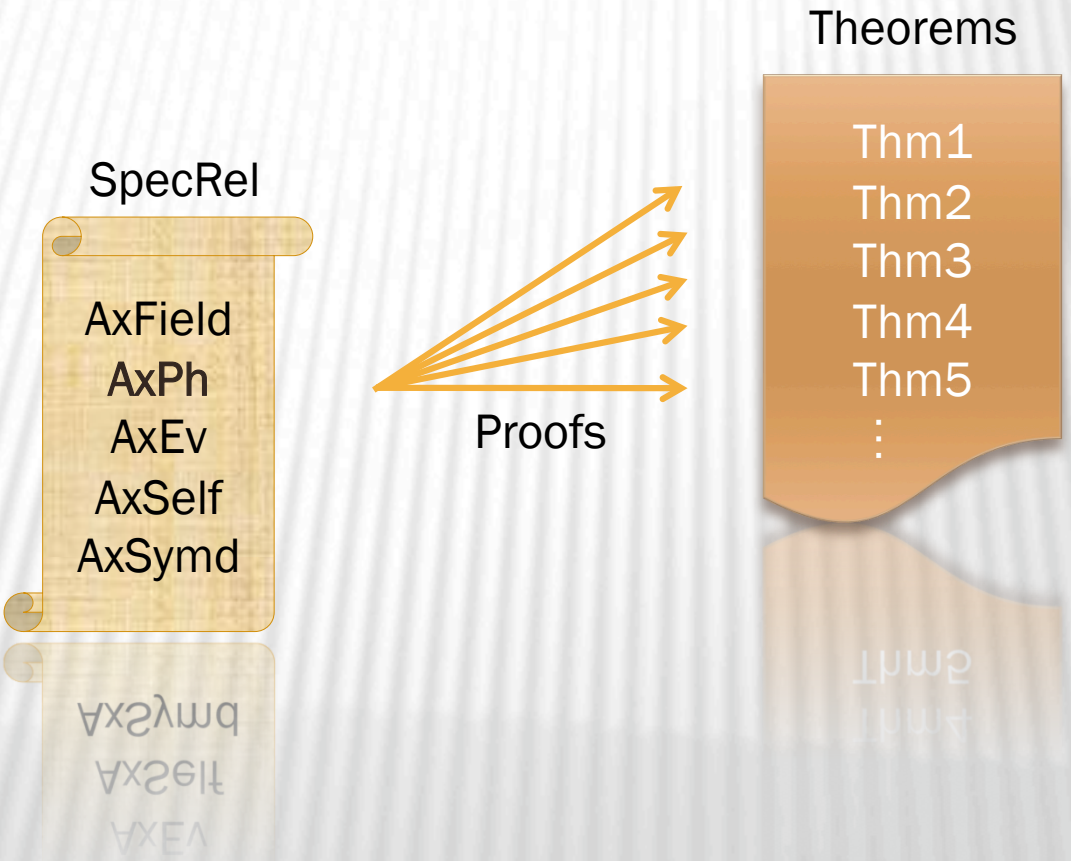
Formalization: $(\forall m, k \in IOb)(\forall p, q, p', q' \in Q^4)$

$$[p_t = q_t \wedge p'_t = q'_t \wedge ev_m(p) = ev_k(p') \wedge ev_m(q) = ev_k(q')] \rightarrow |p_s - q_s| = |p'_s - q'_s|$$

$ev_m(p) = \{b \in B: W(m,p,b)\}$ is the **event** occurring at *p* in *m*'s worldview

SPECREL

SpecRel = {AxField, AxPh, AxEv, AxSelf, AxSymd}



SPECREL

- Thm1

$$SpecRel \vdash (\forall m, k \in IOb)(wline_m(k)\ is\ a\ straight\ line)$$

- Thm2

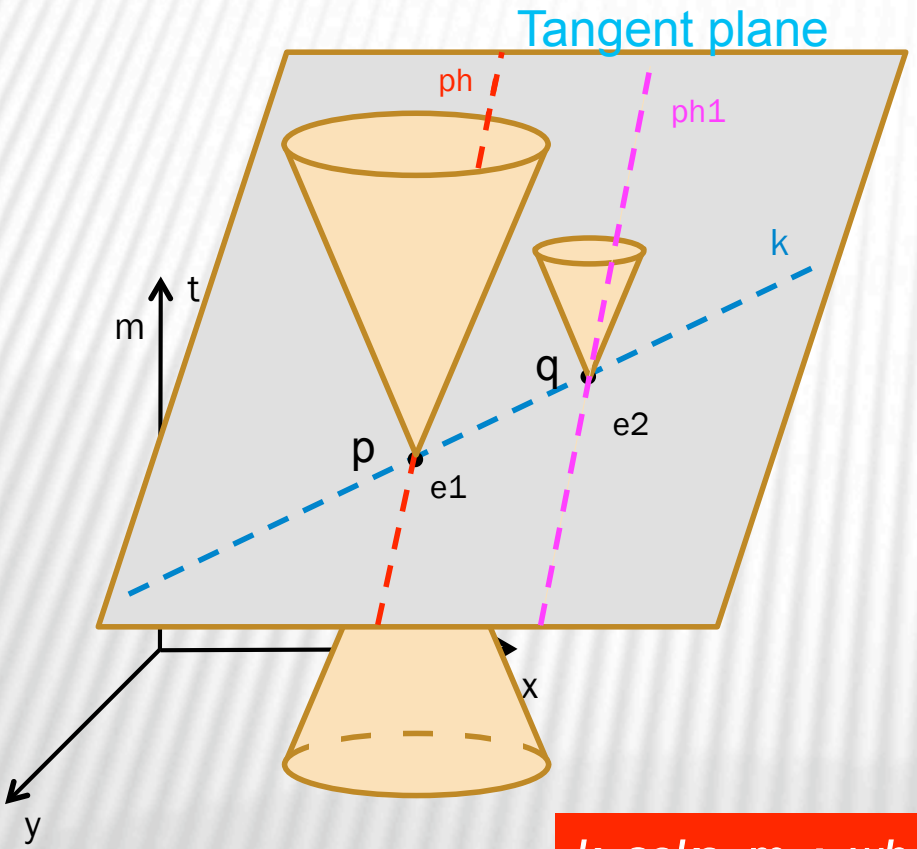
$$SpecRel \vdash NoFTL\ travel$$

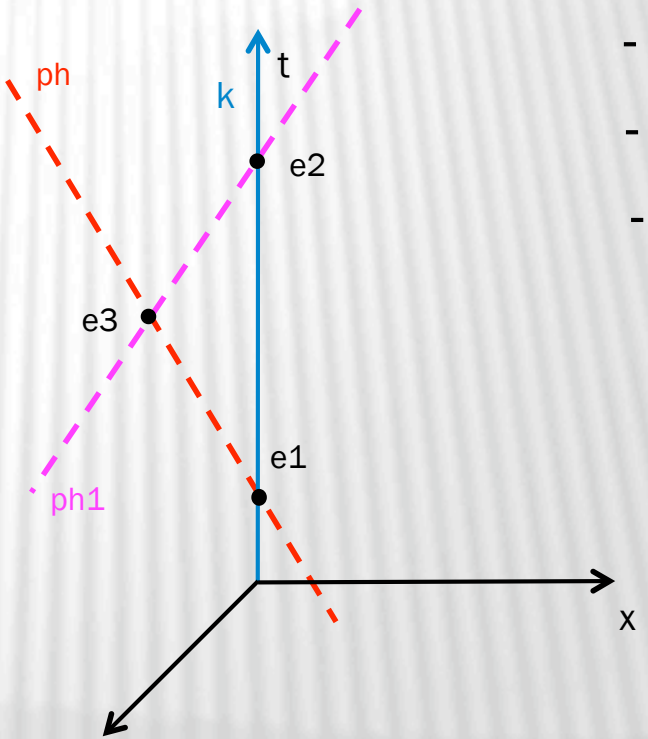
$$NoFTL \overset{df}{\Leftrightarrow} (\forall m, k \in IOb)|v_m(k)| < |v_m(ph)|$$
$$for\ some\ ph \in Ph$$

SPECREL

Proof of Thm2 (NoFTL):

Assume □ ,□ ∈ □□□ ,□,□ ∈ □□□□□ □ □k□, and ä□□ (□)ä> ä□□ (□ℎ′)ä for some □ℎ′





- AxField
- AxPh
- AxSelf
- AxEv

k asks m : where did ph and ph1 meet?

SPECREL

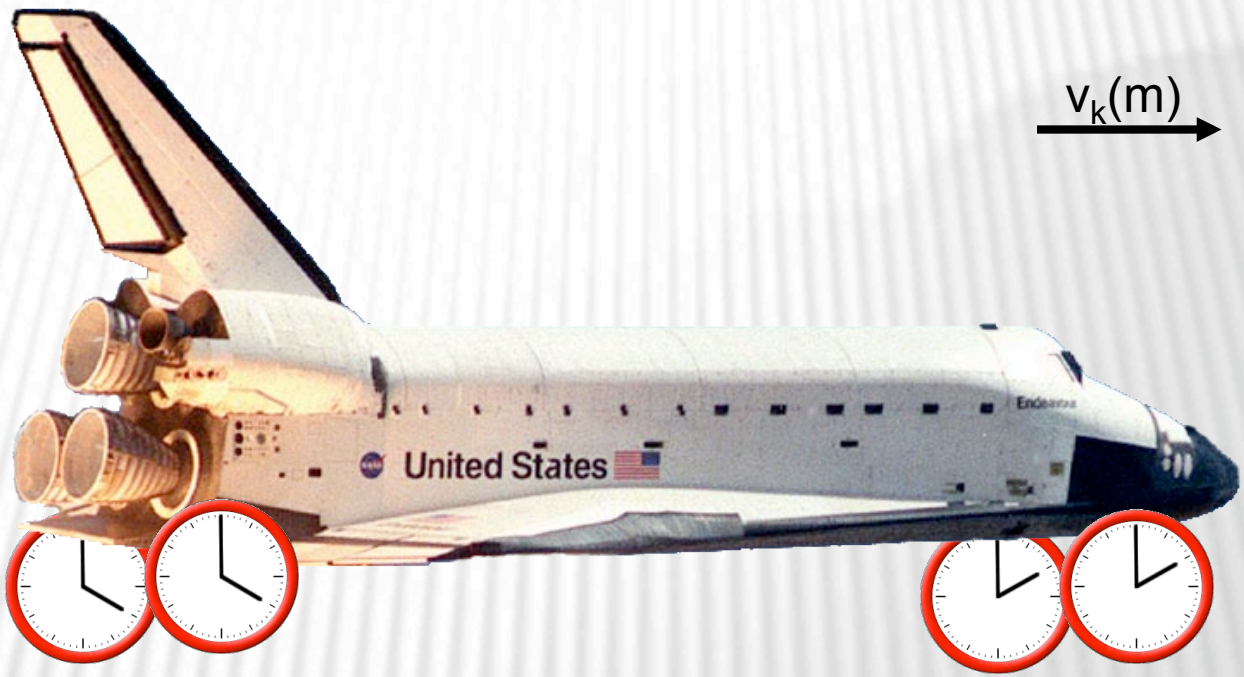
- Thm3
  - SpecRel is consistent
- Thm4
  - No axioms of SpecRel is provable from the rest
- Thm5
  - SpecRel is complete with respect to Minkowski geometries (e.g. implies all the basic paradigmatic effects of Special Relativity - even quantitatively!)

SPECREL

- Thm6
  - SpecRel generates an undecidable theory. Moreover, it enjoys both of Gödel's incompleteness properties
- Thm7
  - SpecRel has a decidable extension, and it also has a hereditarily undecidable extension. Both extensions are physically natural.

RELATIVISTIC EFFECTS

Thm8



- Moving clocks get out of synchronism.

MOVING CLOCKS GET OUT OF SYNCHRONISM

- Thm8 (formalization of clock asynchronism)

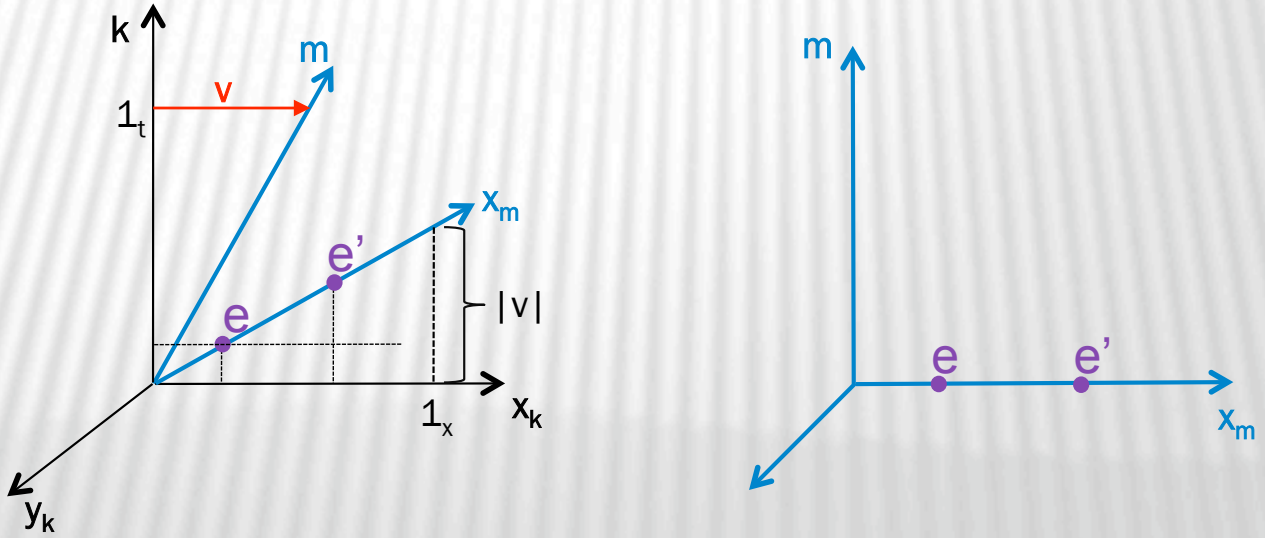
Assume SpecRel. Assume *m,k* ϵ *IOb* and events *e*, *e'* are simultaneous for *m*,

$$i.e.\ \ loc_m(e)_t = loc_m(e')_t$$

(1) Assume *e*, *e'* are separated in the direction of motion of *m* in *k*'s worldview,

$$i.e.\ \ loc_k(e)_s - loc_k(e')_s \parallel v_k(m)$$

$$Then\ \ |loc_k(e)_t - loc_k(e')_t| = |loc_k(e)_s - loc_k(e')_s| \cdot |v_k(m)|$$

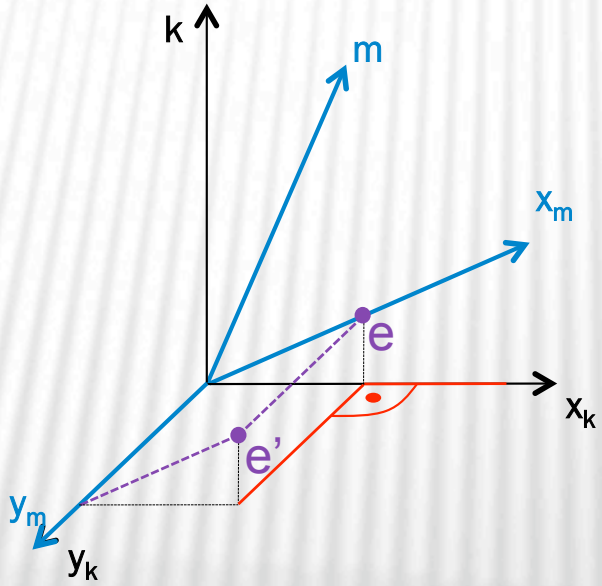


MOVING CLOCKS GET OUT OF SYNCHRONISM

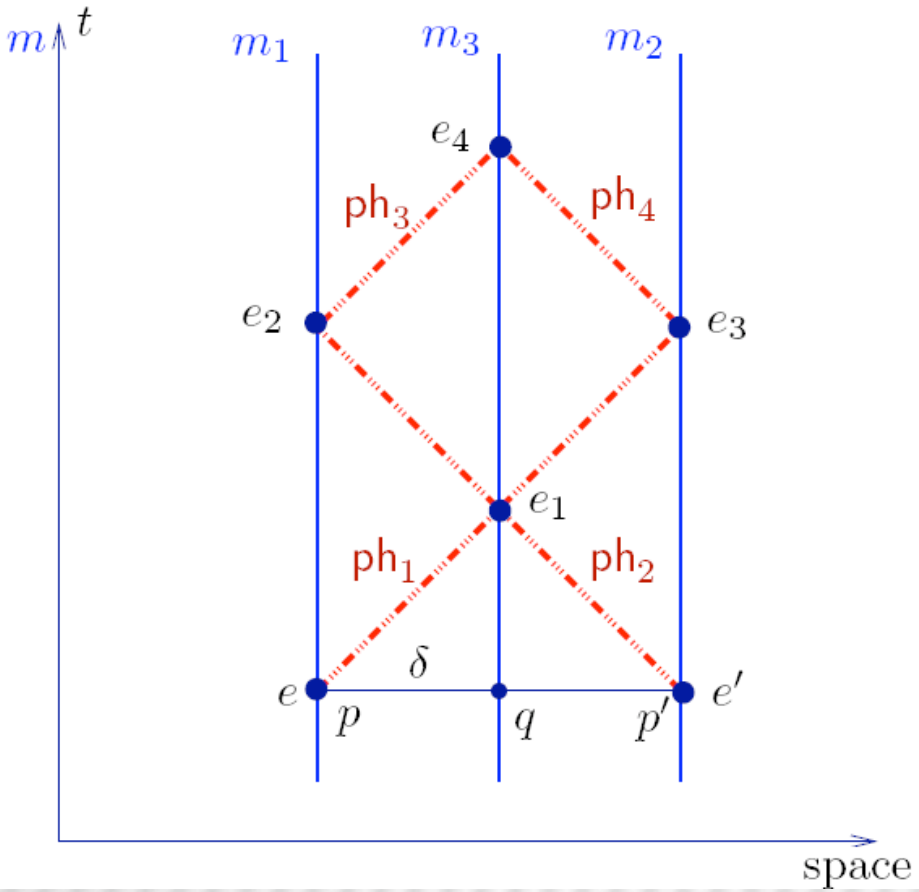
(2) *e*, *e'* are simultaneous for *k*, too

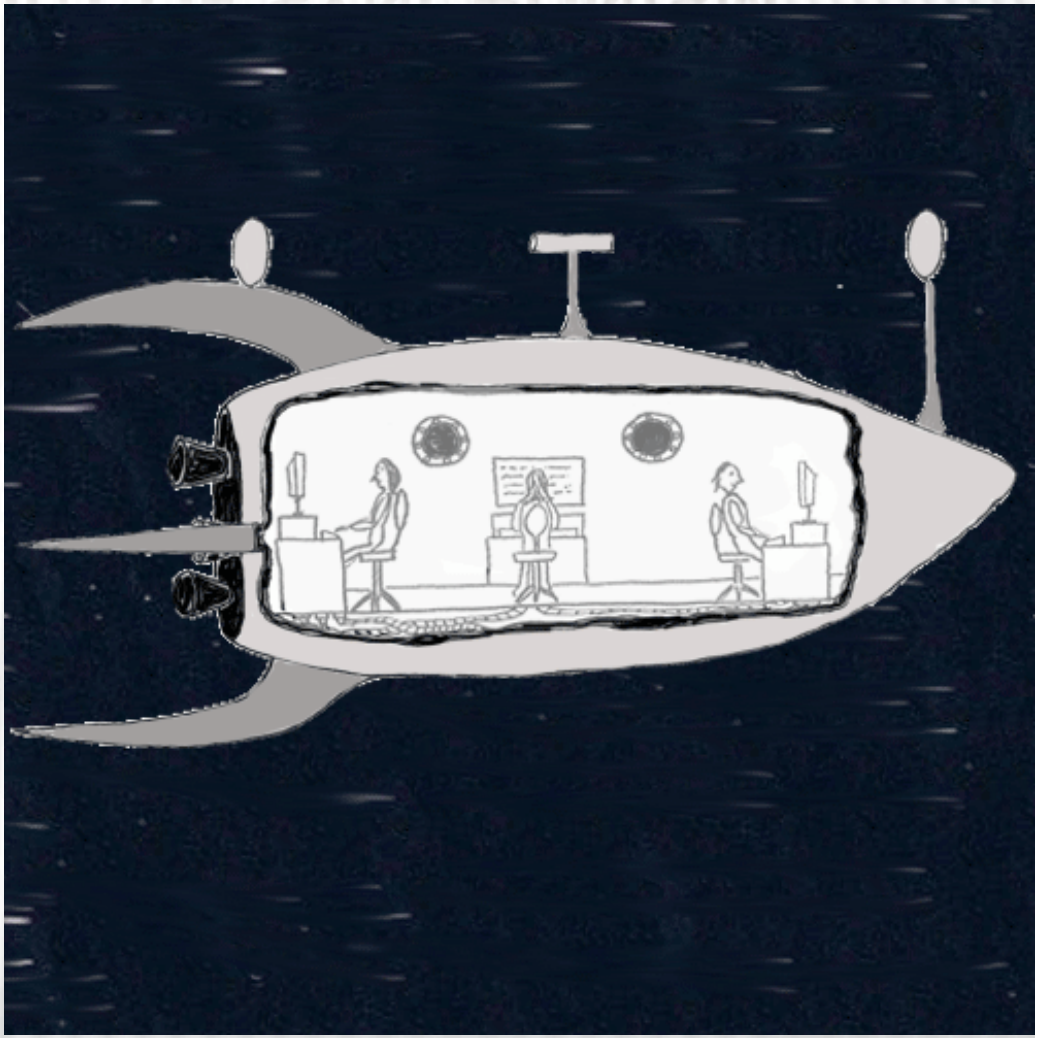
⇔ *e*, *e'* are separated orthogonally to $v_k(m)$ in *k*'s worldview

$$i.e.\ \ loc_k(e)_s - loc_k(e')_s \perp v_k(m)$$



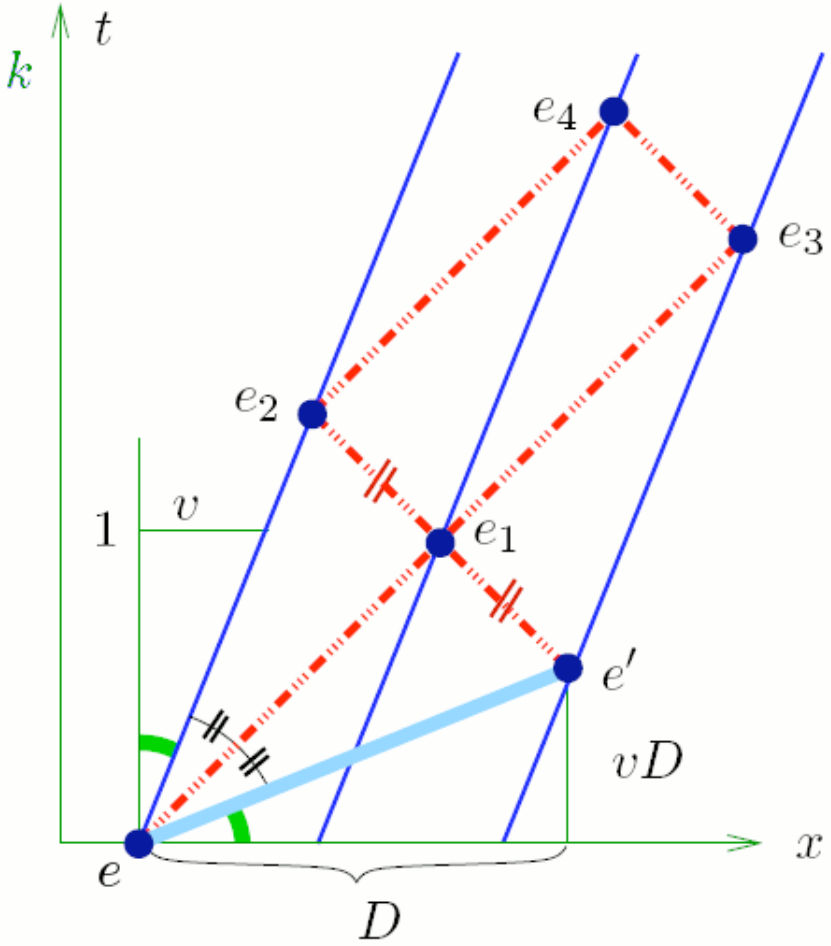
MOVING CLOCKS GET OUT OF SYNCRONISM

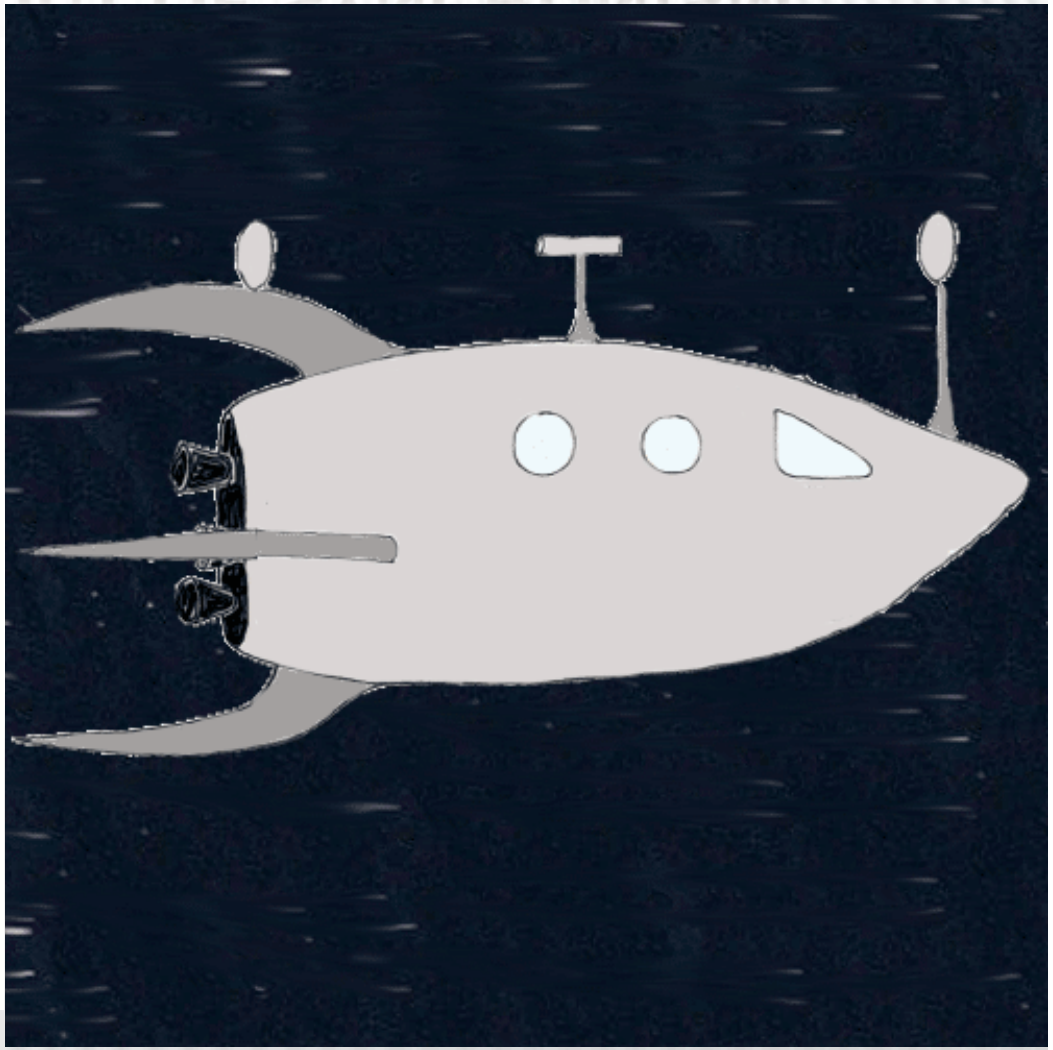




Thought-experiment for proving relativity of simultaneity.

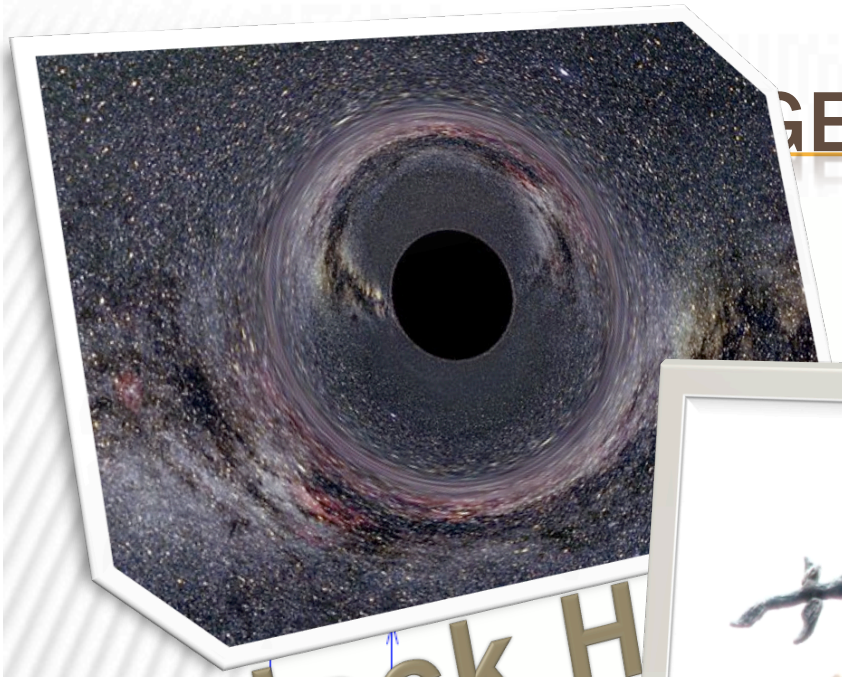
MOVING CLOCKS GET OUT OF SYNCRONISM

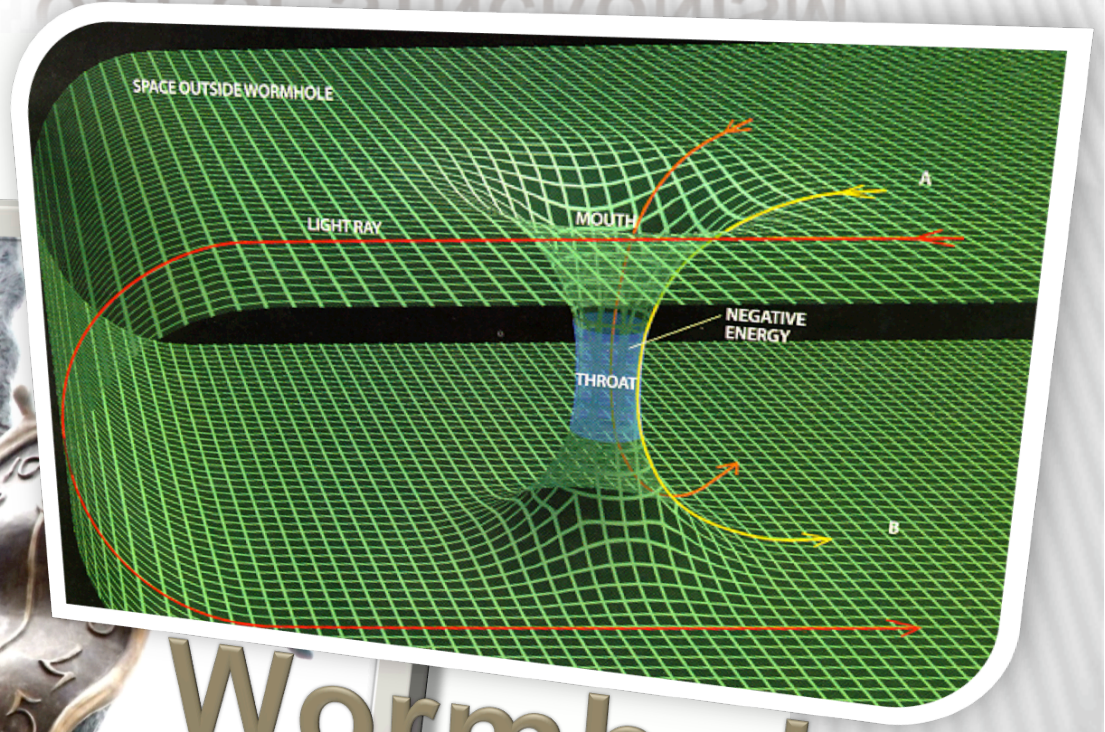




Thought-experiment for proving relativity of simultaneity.

GET OUT OF SYNCRONISM





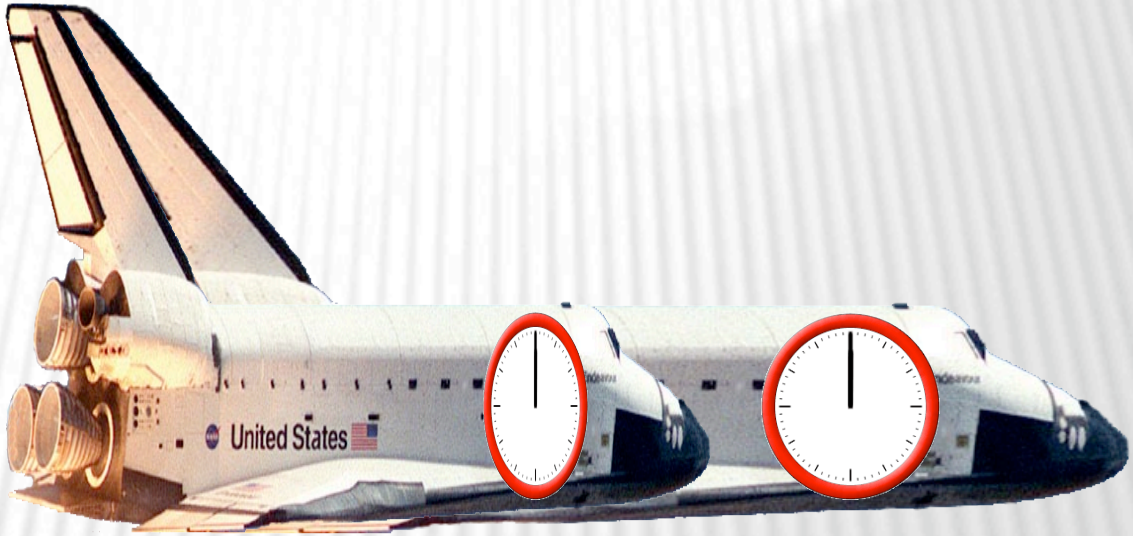
Black H

Wormhole

Timewarp-theory

RELATIVISTIC EFFECTS

Thm9



- Moving clocks slow down
- Moving spaceships shrink

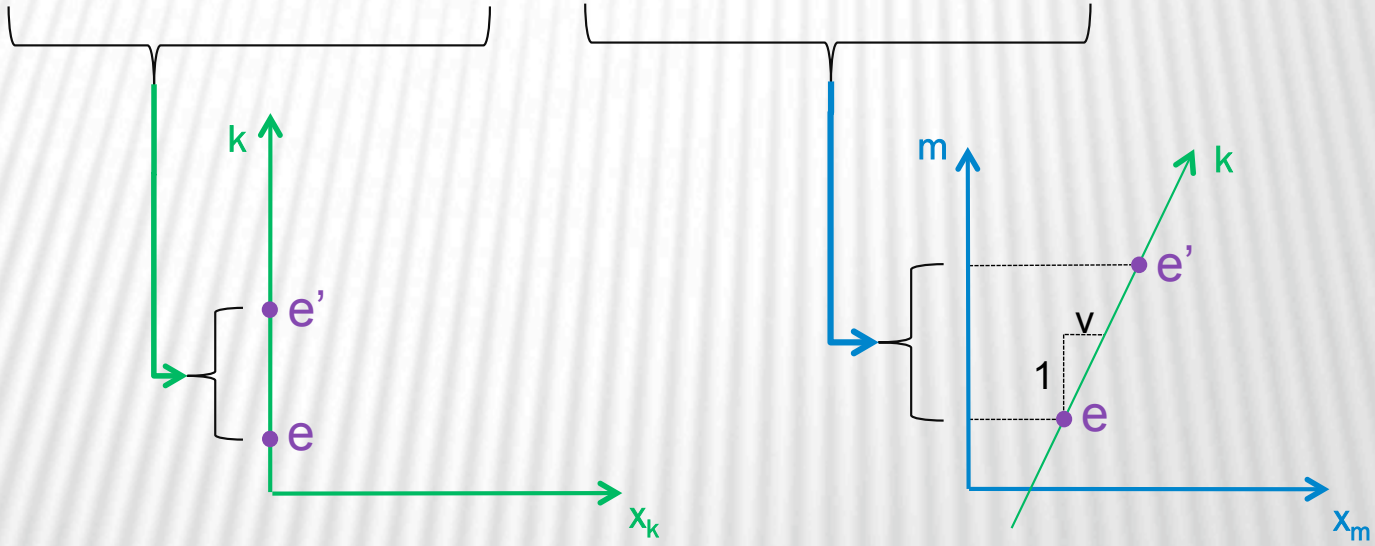
MOVING CLOCKS SLOW DOWN

- Thm9 (formalization of time-dilation)

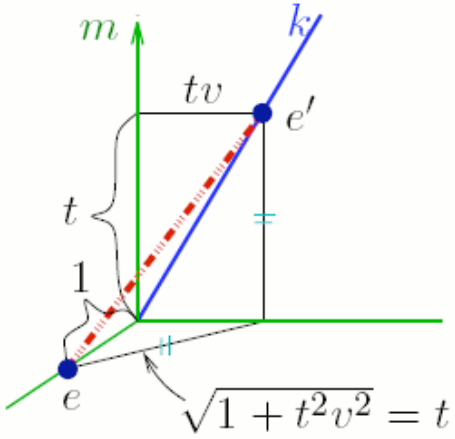
Assume SpecRel. Let *m,k* ϵ *IOb* and events e, e' are on *k*'s lifeline.

$$i.e.\ \ loc_k(e)_s = loc_k(e')_s$$

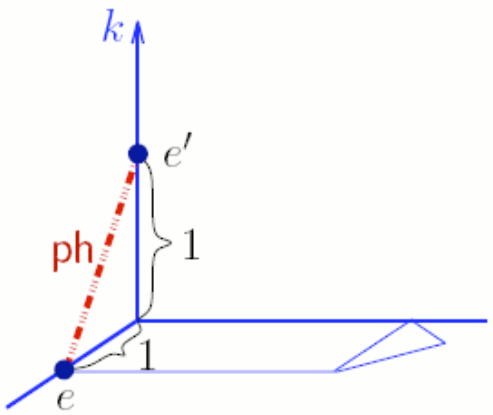
$$Then\ \ |loc_k(e)_t - loc_k(e')_t| = |loc_m(e)_t - loc_m(e')_t| \cdot \sqrt{1 - |v_m(k)|^2}$$



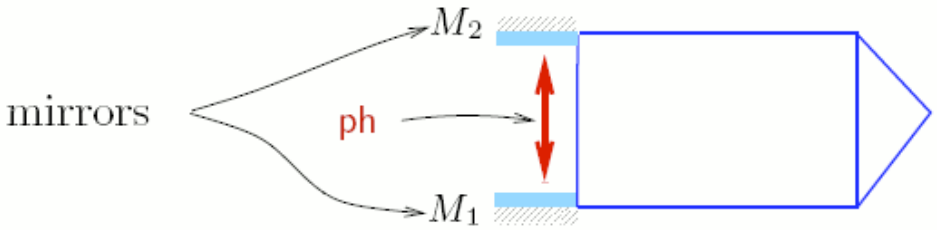
MOVING CLOCKS SLOW DOWN



m's worldview



k's worldview



Einstein's light-clock

Thought-experiment proving time-dilation (Einstein's light-clock).

MOVING SPACESHIPS SHRINK

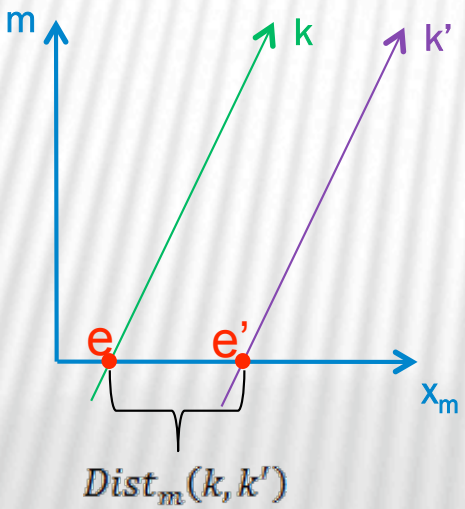
- Thm10 (formalization of spaceship shrinking)

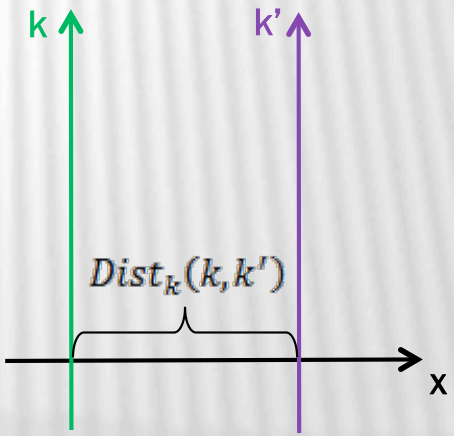
  Assume SpecRel. Let *m,k,k'* ϵ *IOb* and assume $v_k(k') = 0.$

$$Dist_m(k,k') \coloneqq |loc_m(e) - loc_m(e')|$$

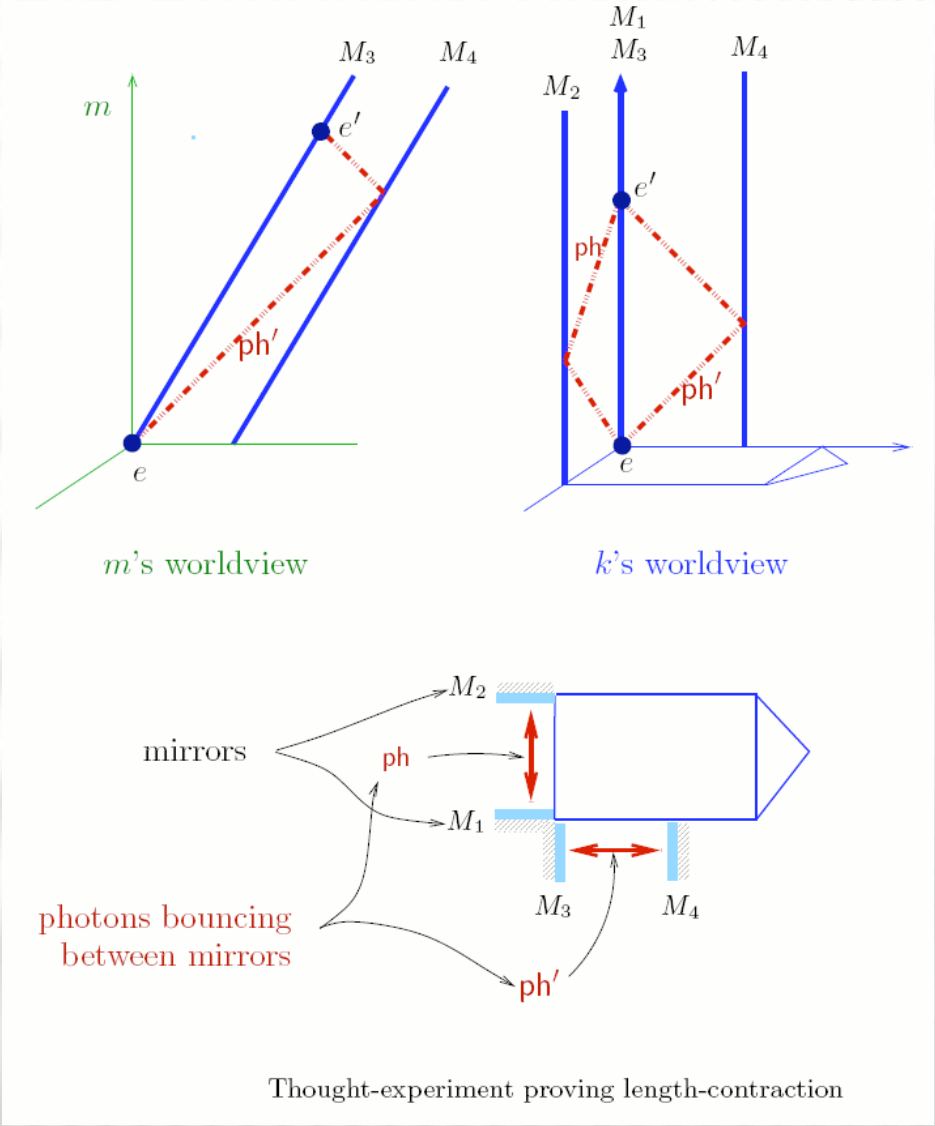
$$where\ \ loc_k(e)_s = loc_k(e')_s = 0\ \ and\ loc_m(e)_t - loc_m(e')_t$$

$$Dist_m(k,k') = \sqrt{1 - |v_m(k)|^2} \cdot Dist_k(k,k')$$



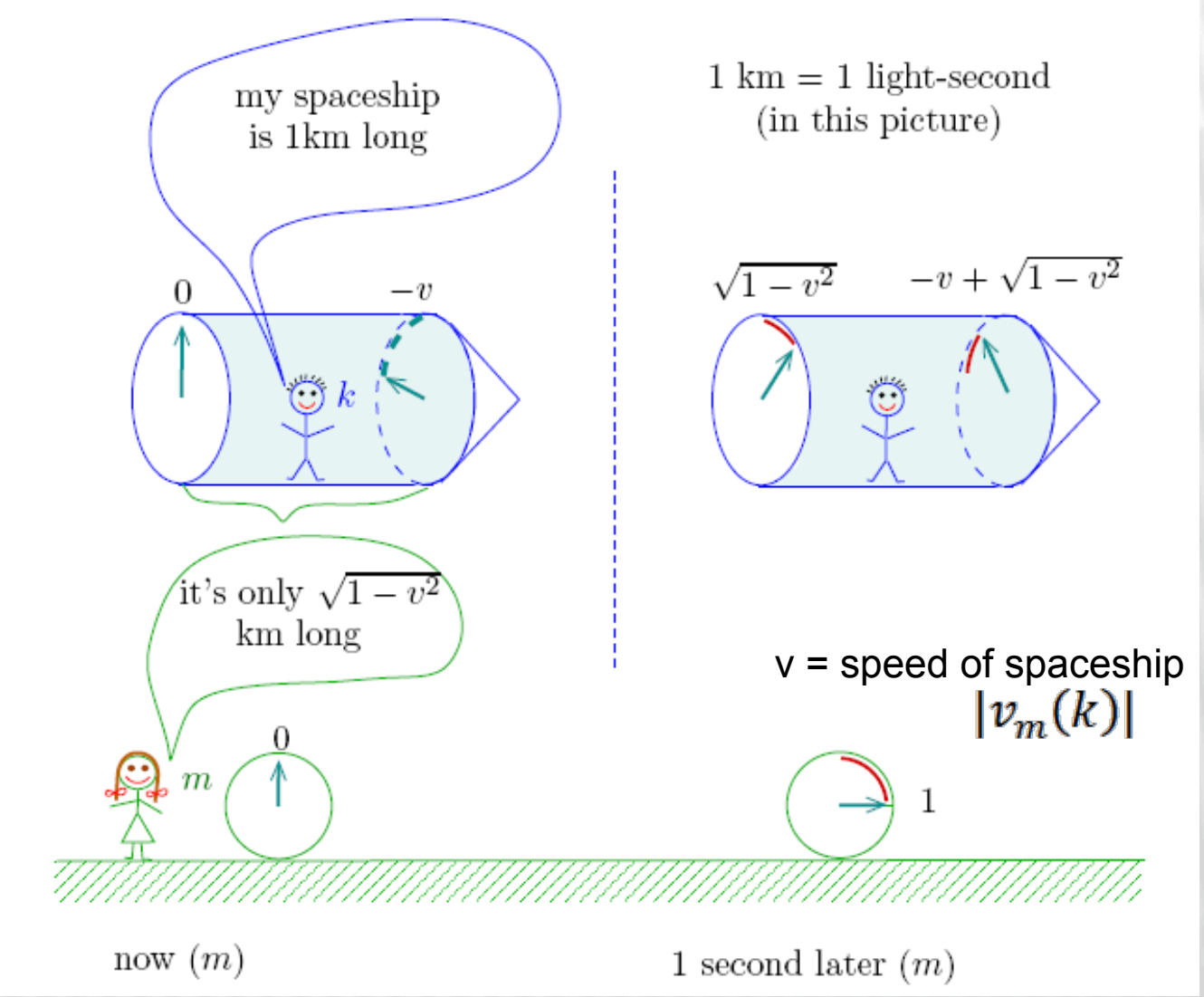


MOVING SPACESHIPS SHRINK

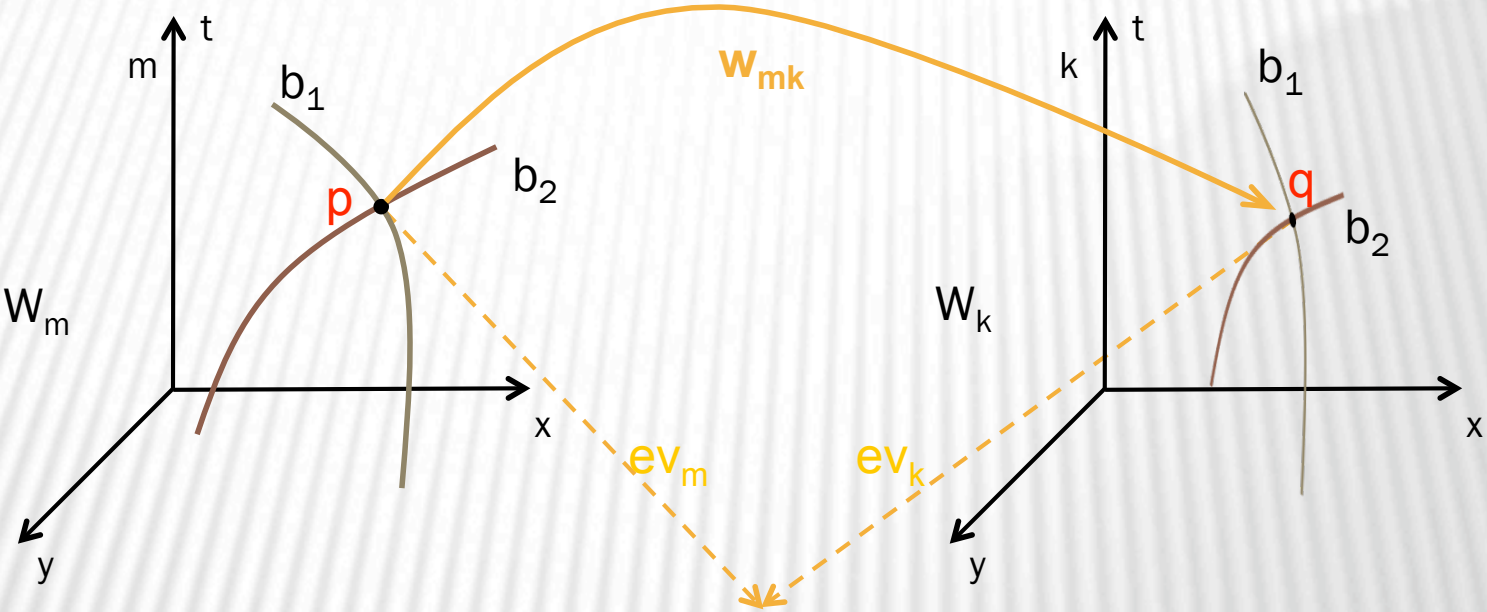


Thought-experiment proving length-contraction

RELATIVISTIC EFFECTS



WORLDVIEW TRANSFORMATION

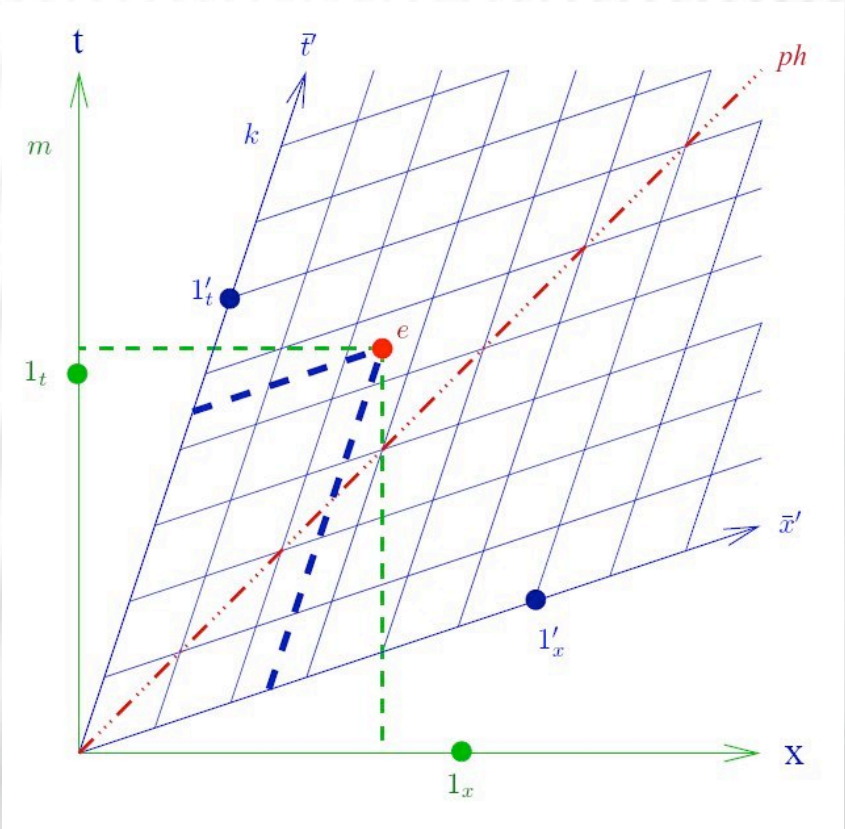


$$w_{mk} := \{\langle p, q\rangle : ev_m(p) = ev_k(q)\}$$

SPECREL

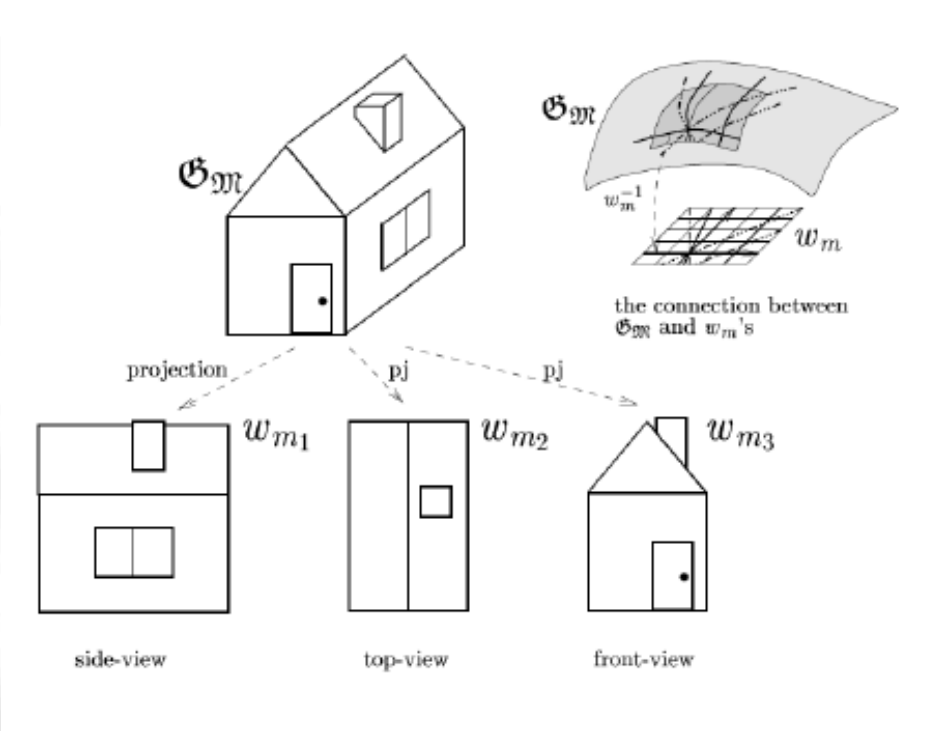
- Thm11

  SpecRel ⊢ The worldview transformations w_{mk} are Lorentz transformations (composed perhaps with a translation).



OTHER FORMALISATIONS OF SPECREL

Minkowskian Geometry:



Hierarchy of theories. Interpretations between them. Definitional equivalence. Contribution of relativity to logic: definability theory with new objects definable (and not only with new relations definable). J. Madarasz's dissertation.

SPECREL

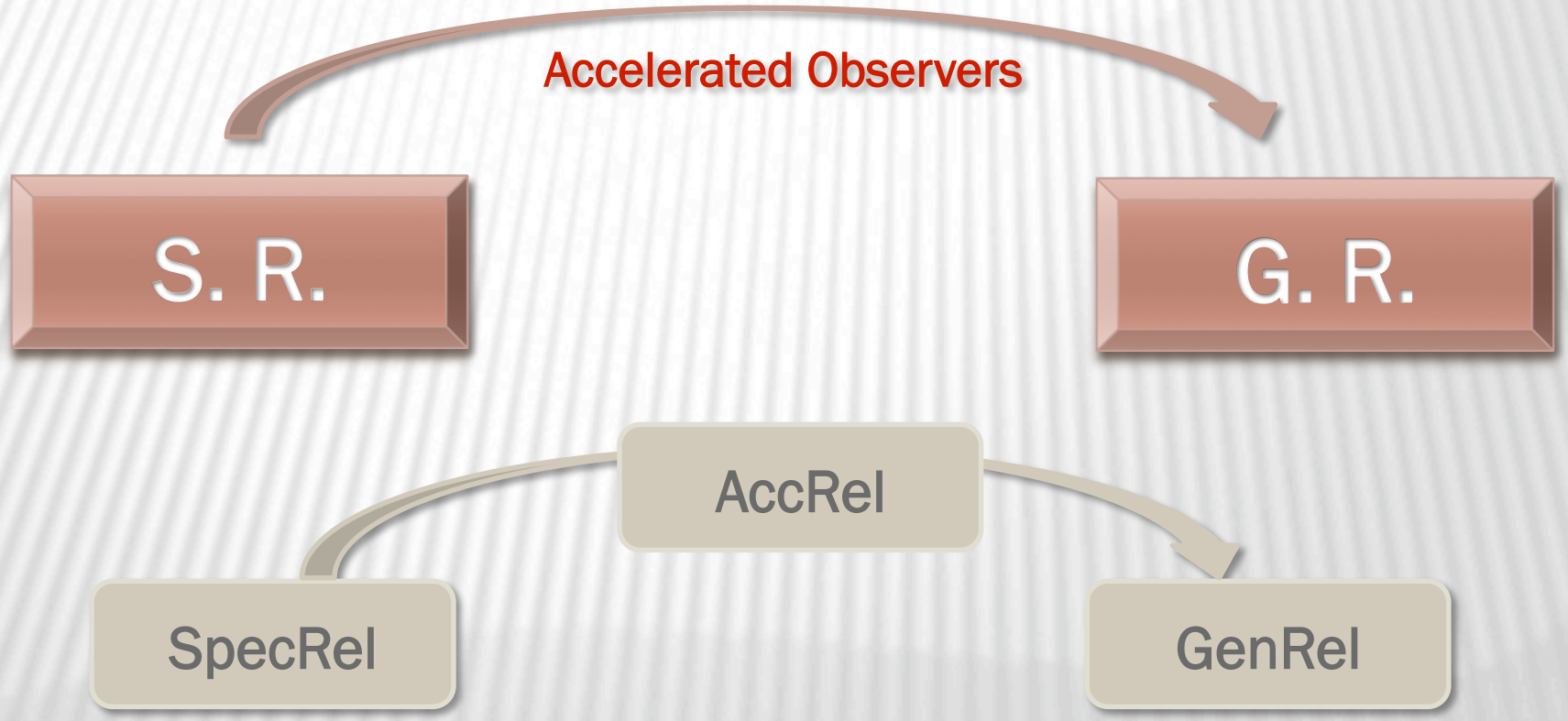
Conceptual analysis of SR goes on ... on our homepage

New theory is coming:

THEORY OF ACCELERATED OBSERVERS

AccRel = SpecRel + AccObs.

AccRel is stepping stone to GenRel via Einstein's EP



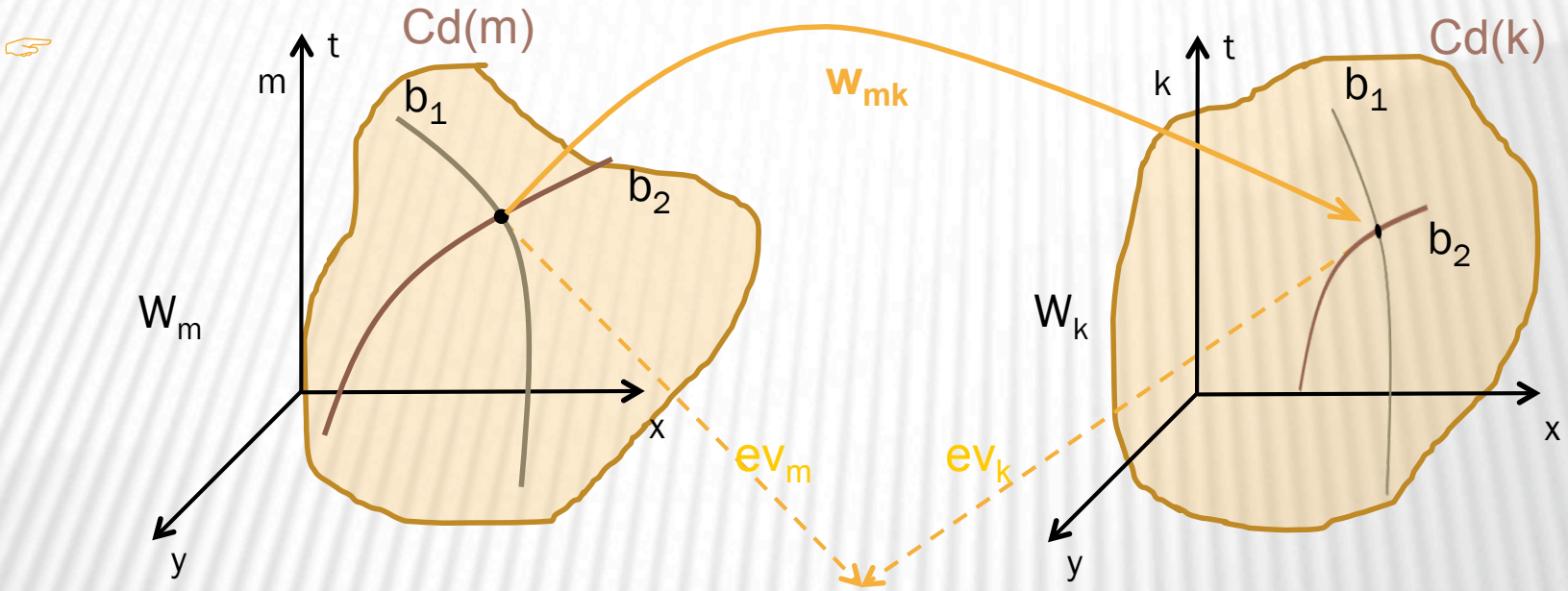
LANGUAGE FOR ACCREL

The same as for SpecRel.

Recall that $W \subseteq B \times Q^4 \times B$

$$Ob := Dom(W) := \{k \in B : \exists pb \ \ W(k, p, b)\}$$

LANGUAGE FOR ACCREL



$$Cd(m) := \{p : ev_m(p) \cap Ob \neq \emptyset\}$$

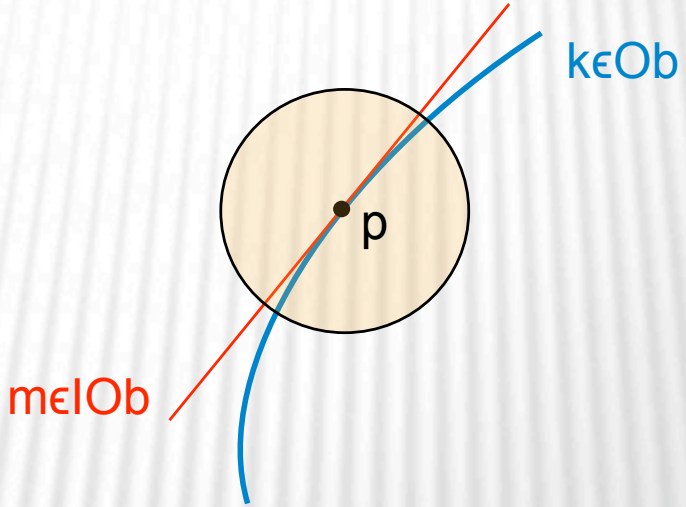
$$w_{mk} := \{\langle p, q\rangle : p \in Cd(m) \wedge ev_m(p) = ev_k(q)\}$$

World-view transformation

AXIOMS FOR ACCELERATED OBSERVERS

- AxCmv

  At each moment p of his worldline, the accelerated observer k "sees" (=coordinatizes) the nearby world for a short while as an inertial observer m does, i.e. "the linear approximation of w_{mk} at p is the identity".



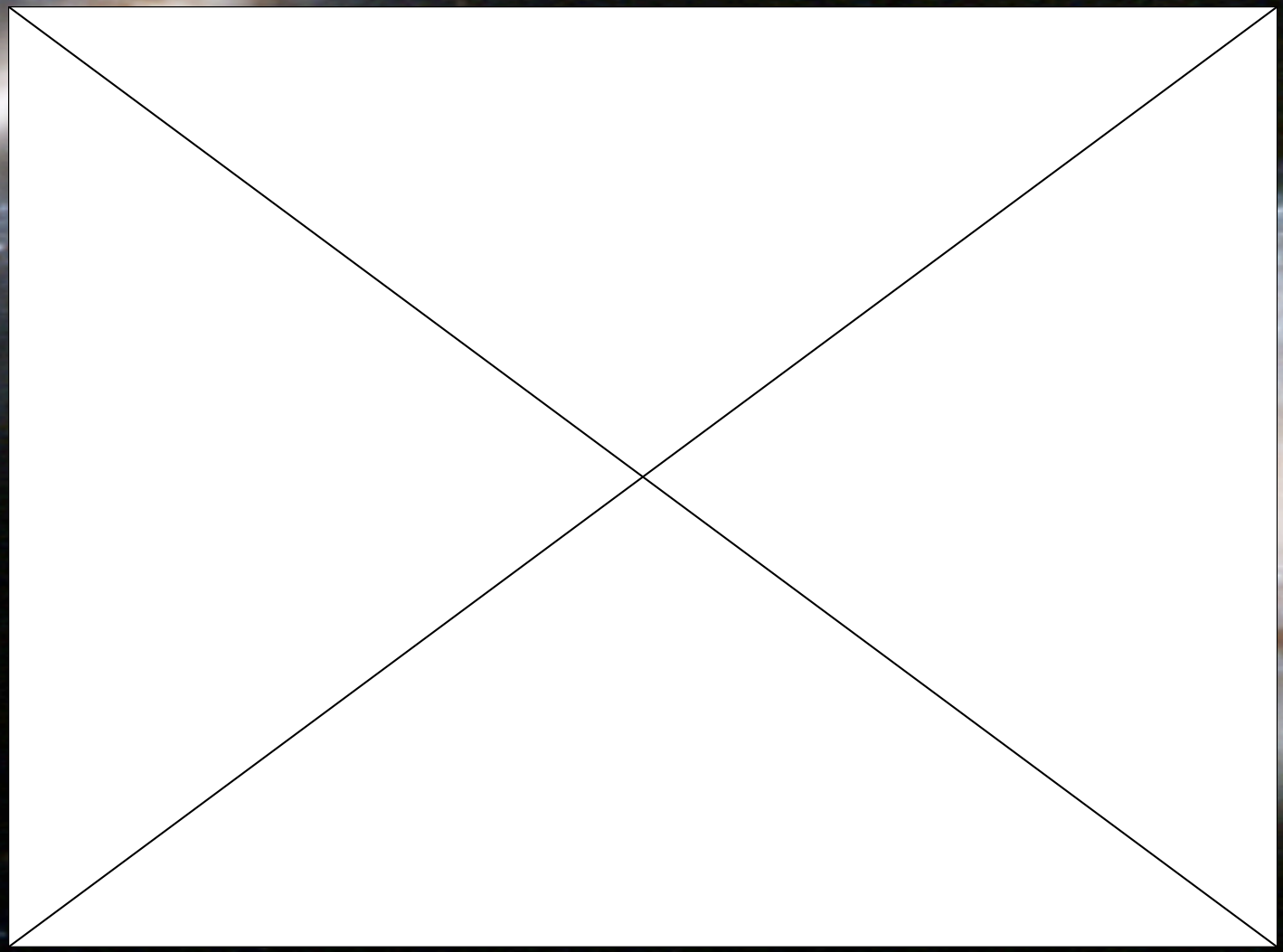
Formalization:

$$\forall k \in Ob\ \forall p \in wline_k(k) \exists m \in IOb\ Dif(w_{mk})(p) = Id$$

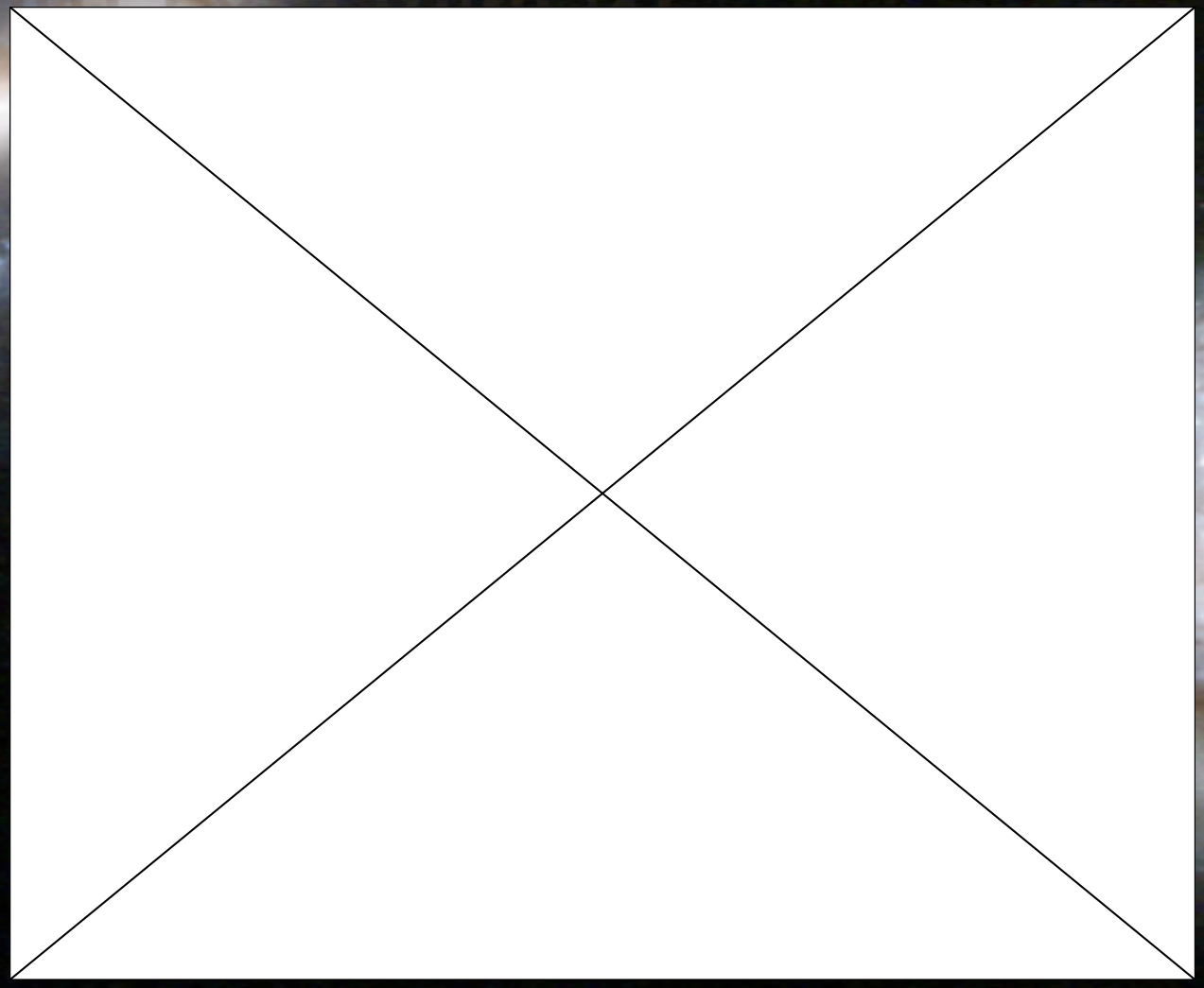
Let *F* be an affine mapping (definable).

$$Dif(w_{mk})(p) = F \overset{def}{\iff} \forall \varepsilon > 0\ \exists \delta > 0\ \forall q(|q-p| \leq \delta \Rightarrow |w_{mk}(q) - F(q)| \leq \varepsilon \cdot |q-p|)$$

THOUGHT EXPERIMENT



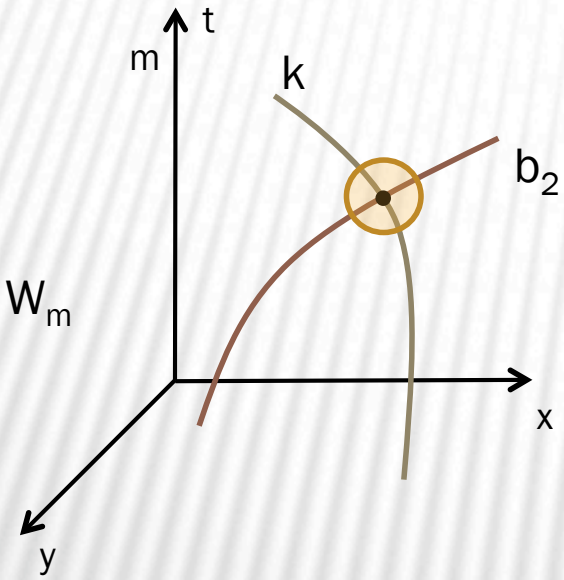
THOUGHT EXPERIMENT

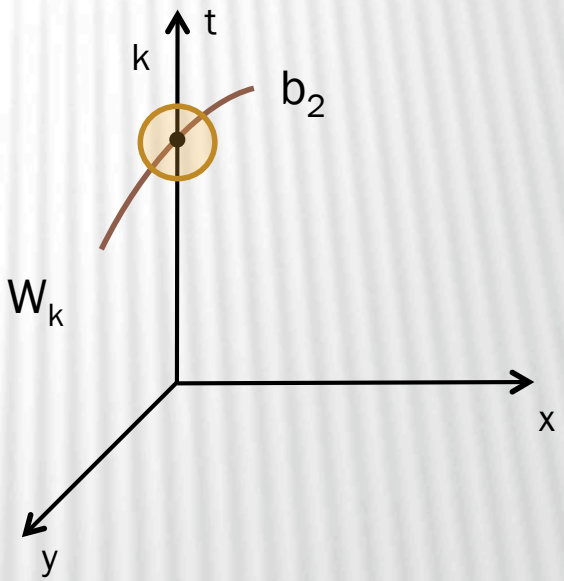


AXIOMS FOR ACCELERATED OBSERVERS

AxEv$^-$

If m "sees" k participate in an event, then k cannot deny it.





Formalization:

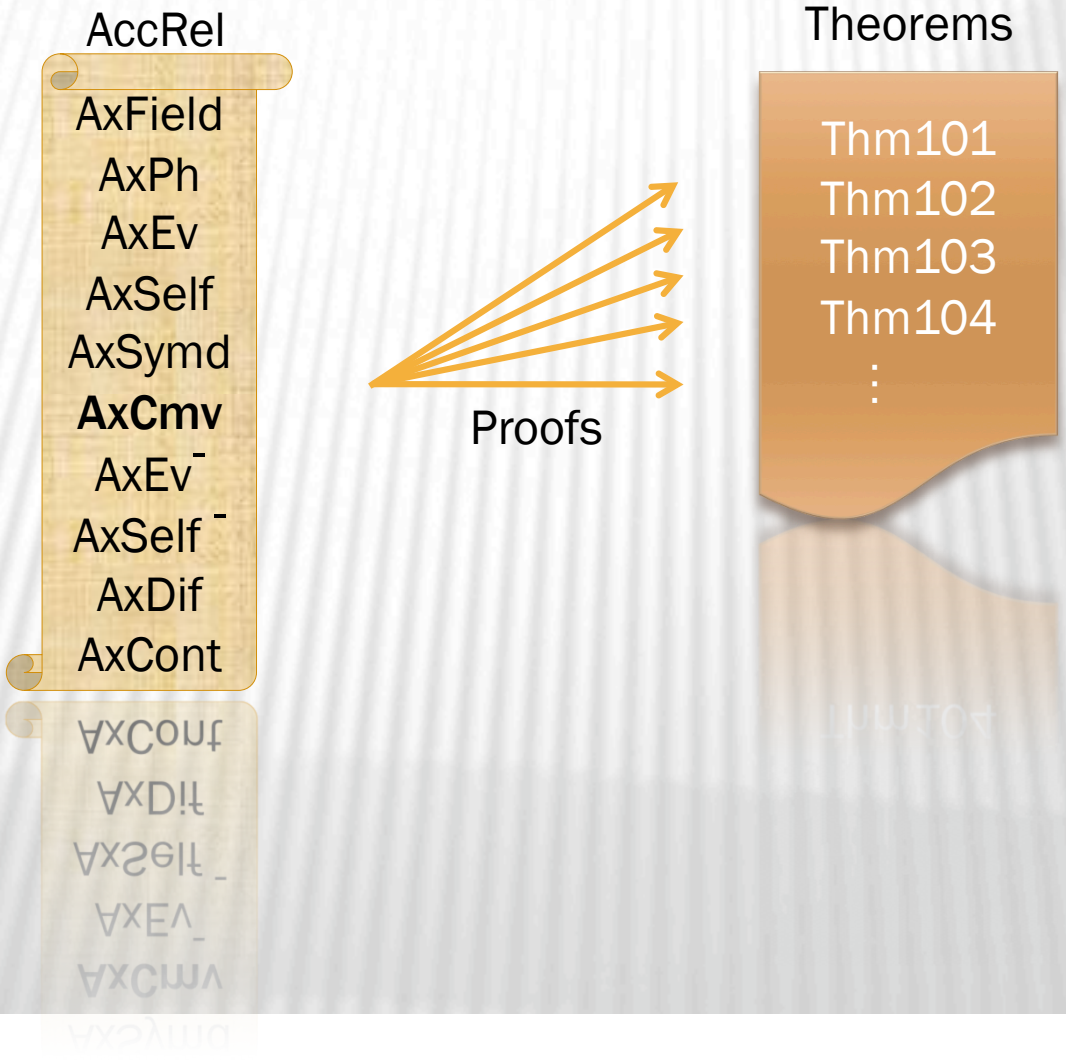
$$(\forall m, k \in Ob)\ wline_m(k) \subseteq Dom(w_{mk})$$

AXIOMS FOR ACCELERATED OBSERVERS

- AxSelf ⁻
  - The world-line of any observer is an open interval of the time-axis, in his own world-view
- AxDif
  - The worldview transformations have linear approximations at each point of their domain (i.e. they are differentiable).
    $\forall m, k \in Ob \ \ \forall p \in Dom(w_{mk}) \ \exists \text{ affine } F \ Dif(w_{mk})(p) = F$
- AxCont
  - Bounded definable nonempty subsets of Q have suprema. Here “definable” means “definable in the language of AccRel, parametrically”.

ACCREL

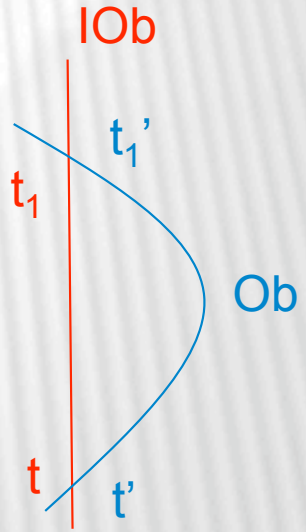
$$AccRel = SpecRel + AxCmv + AxEv^{-} + AxSelf^{-} + AxDif + AxCont$$



ACCREL

- Thm101

$$AccRel \vdash \text{“}twin\ paradox\text{”}$$
$$AccRel - AxCont \nvdash \text{“}twin\ paradox\text{”}$$

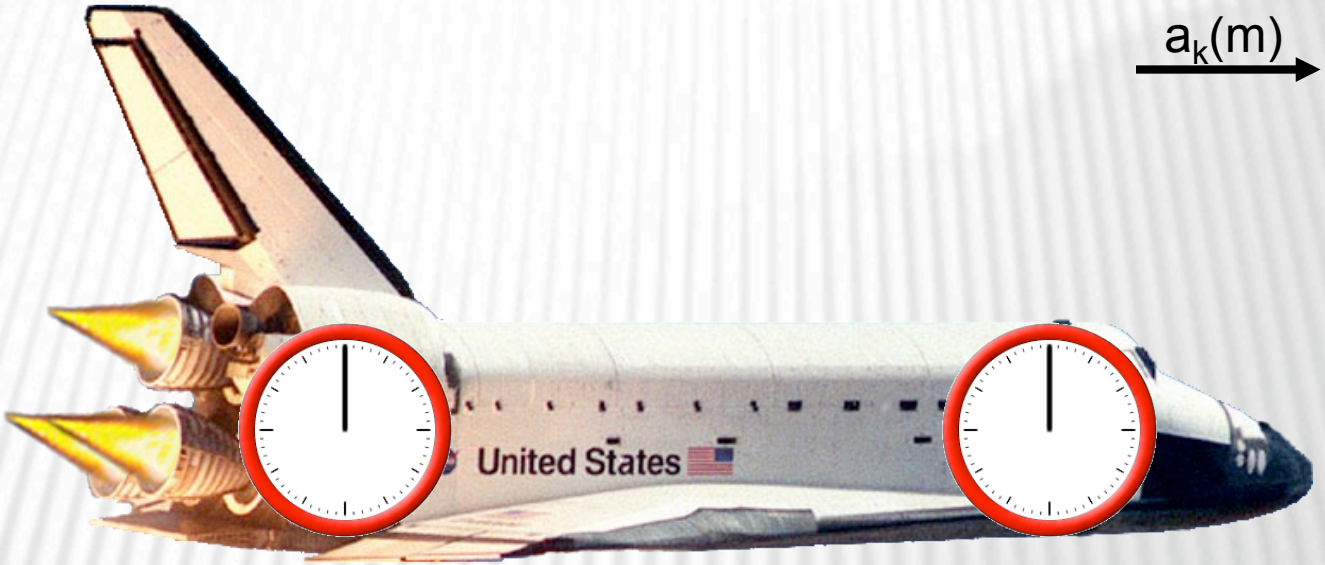


$$AccRel \vdash (\forall m \in IOb)(\forall k \in Ob)$$
$$[ev_m(t,\bar{0}) = ev_k(t',\bar{0}) \neq ev_k(t_1{}',\bar{0}) = ev_m(t_1,\bar{0})\ \&\ (\exists t < t_2 < t_1)\ k \notin ev_m(t_2,\bar{0})]$$
$$\rightarrow |t_1 - t| > |t_1' - t'|$$

Appeared: Found. Phys. 2006, Madarász, J. X., Németi, I., Székely, G.

ACCREL

Thm102



- Acceleration causes slow time.

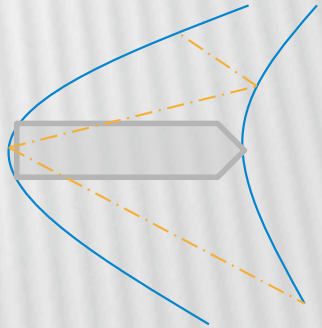
ACCREL

- Thm102

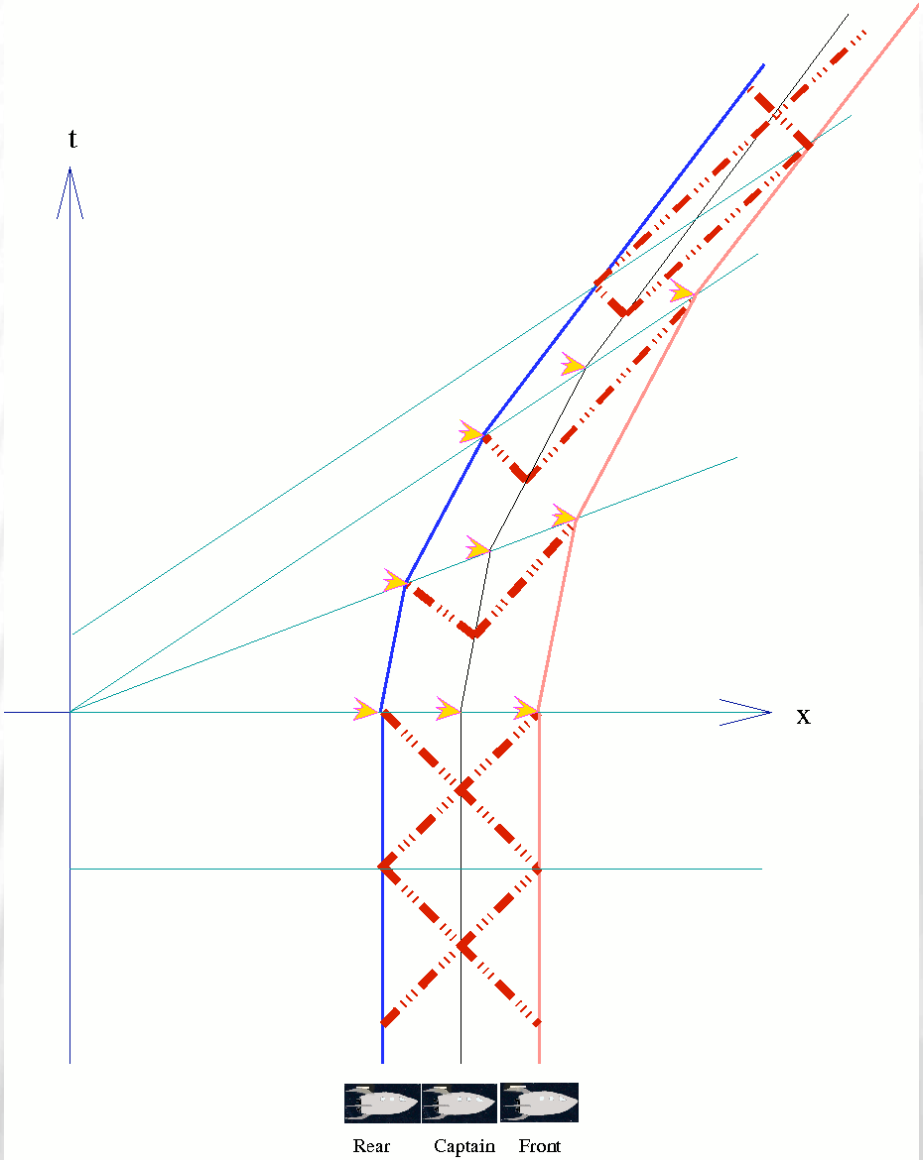
$$AccRel \vdash \text{“}gravitation\ causes\ slow\ time\text{”}$$

  - I.e., clocks at the bottom of spaceship run slower than the ones at the nose of the spaceship, both according to nose and bottom (watching each other by radar).

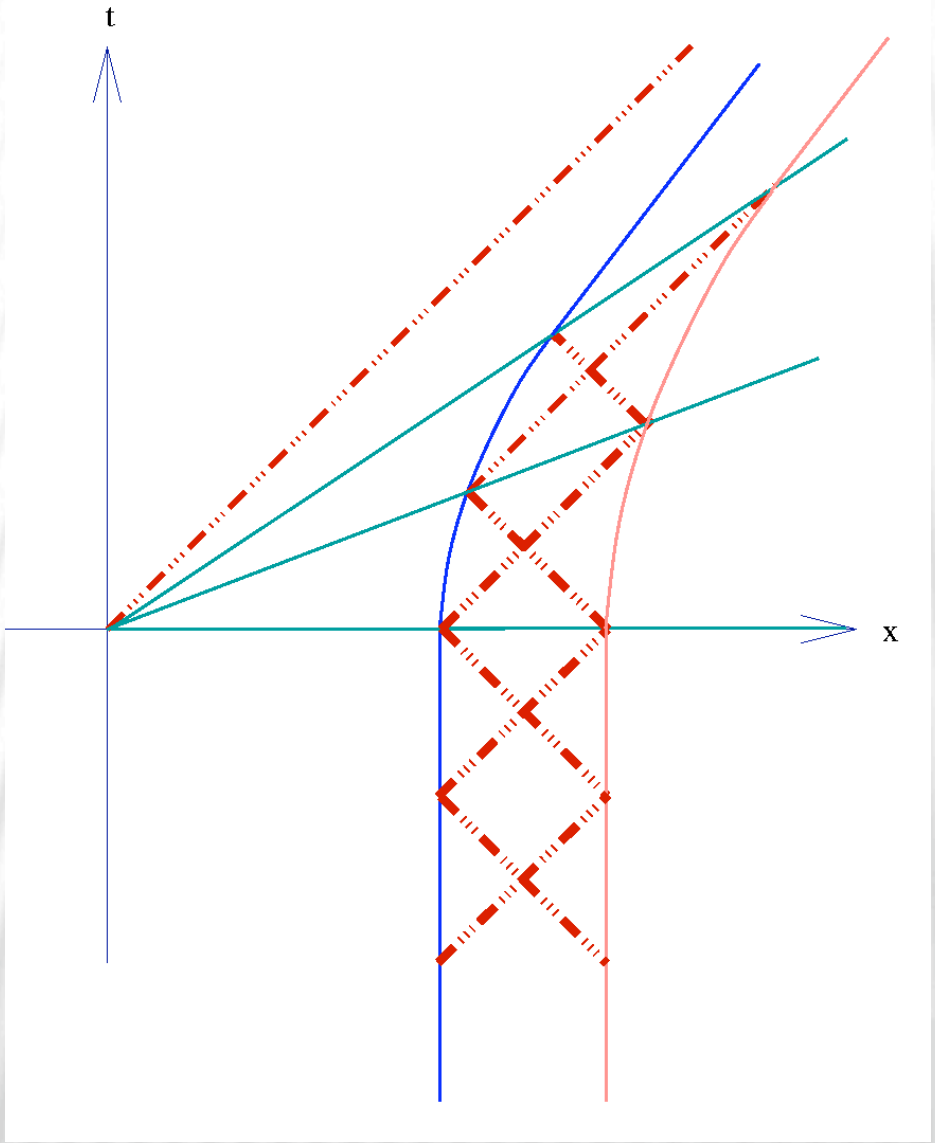
Appeared: Logic for XXIth Century. 2007, Madarász, J. X., Németi, I., Székely, G.



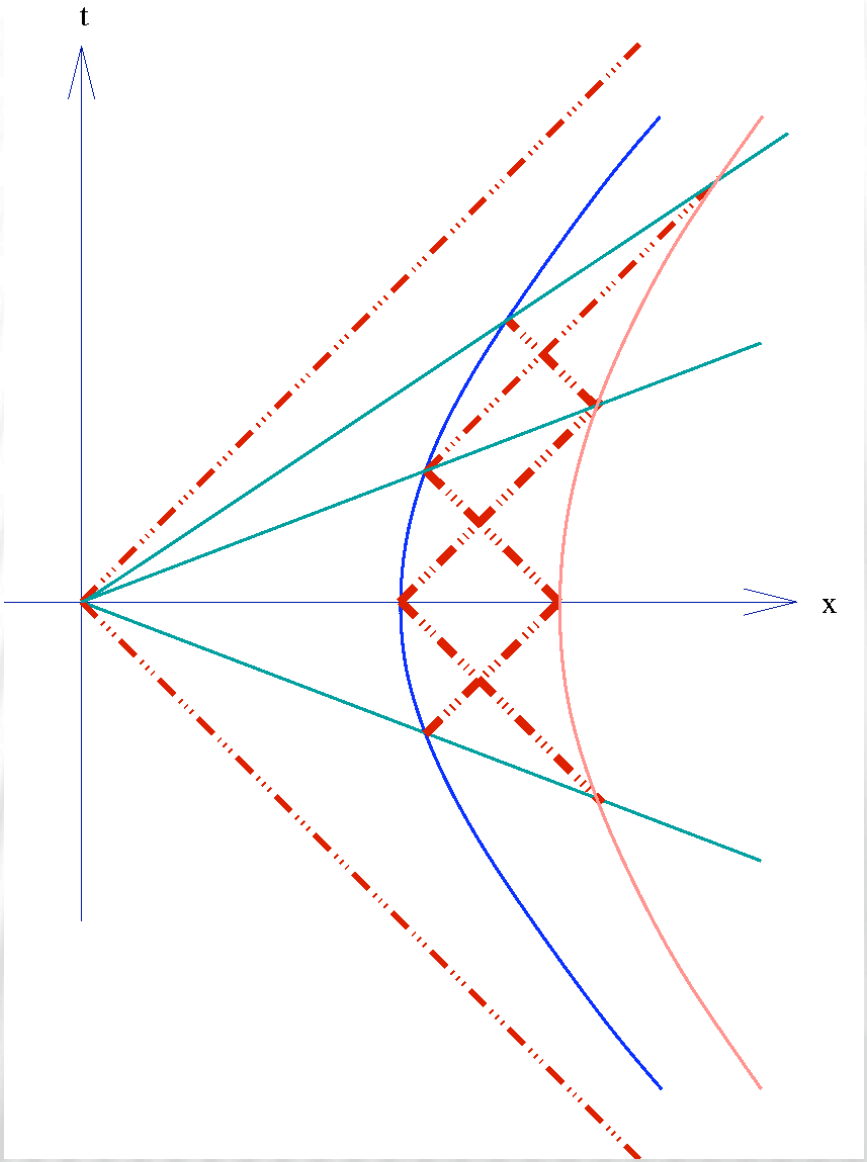
ACCELERATING SPACEFLEET



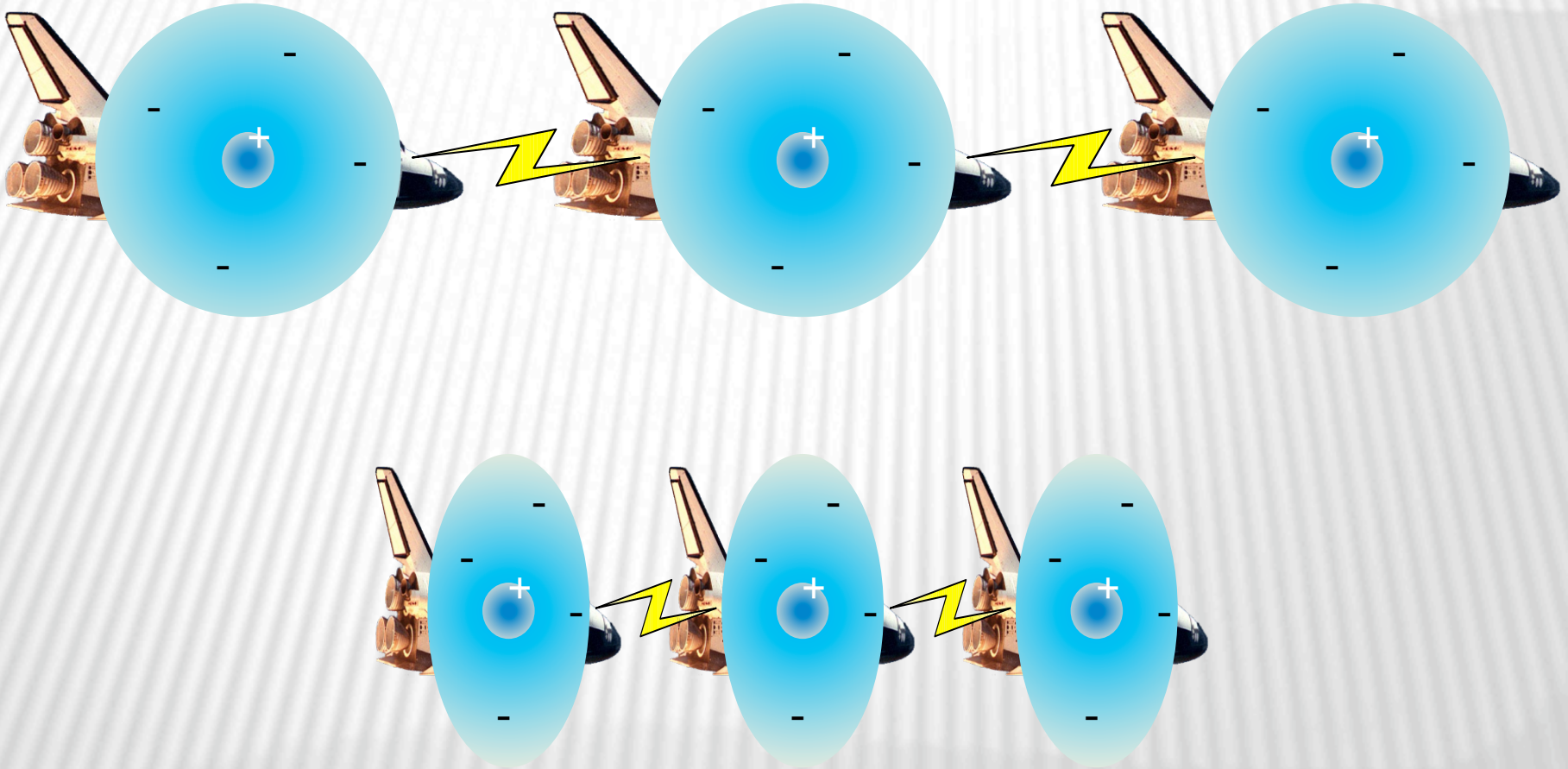
ACCELERATING SPACEFLEET



ACCELERATING SPACEFLEET



ARE THE EFFECTS REAL?



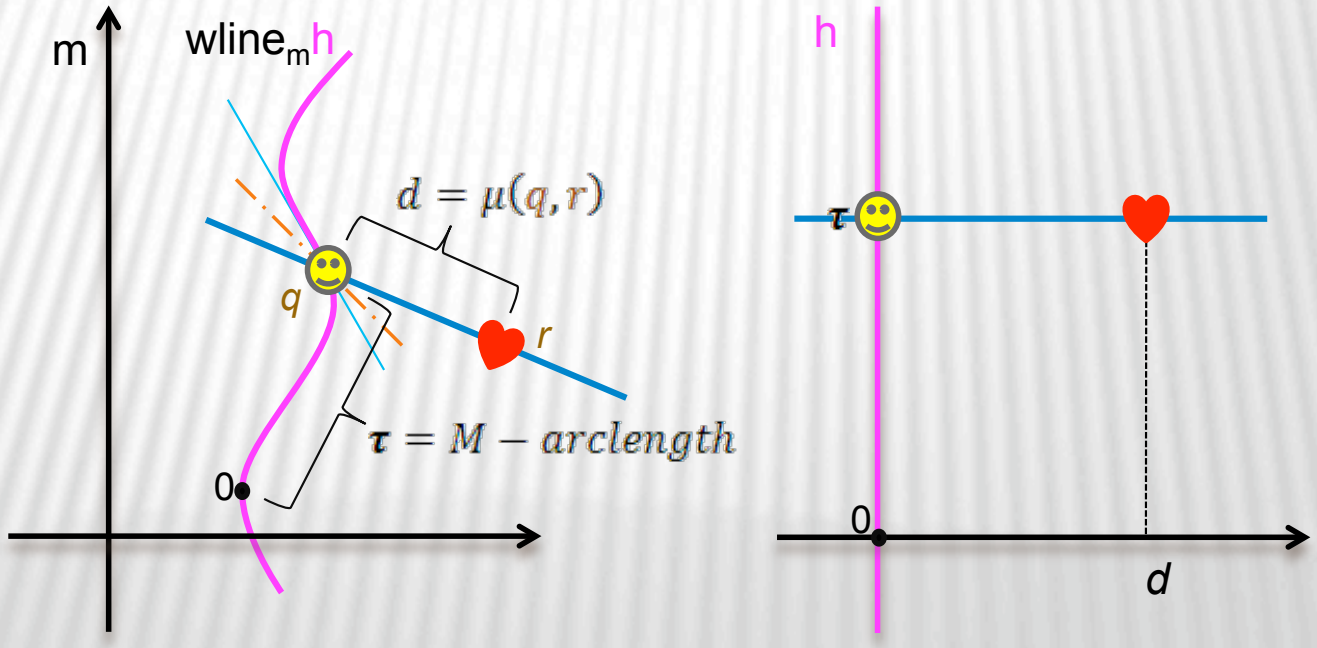
ACCREL

- Thm103

$$AccRel + (Ob \neq IOb) \text{ is consistent.}$$

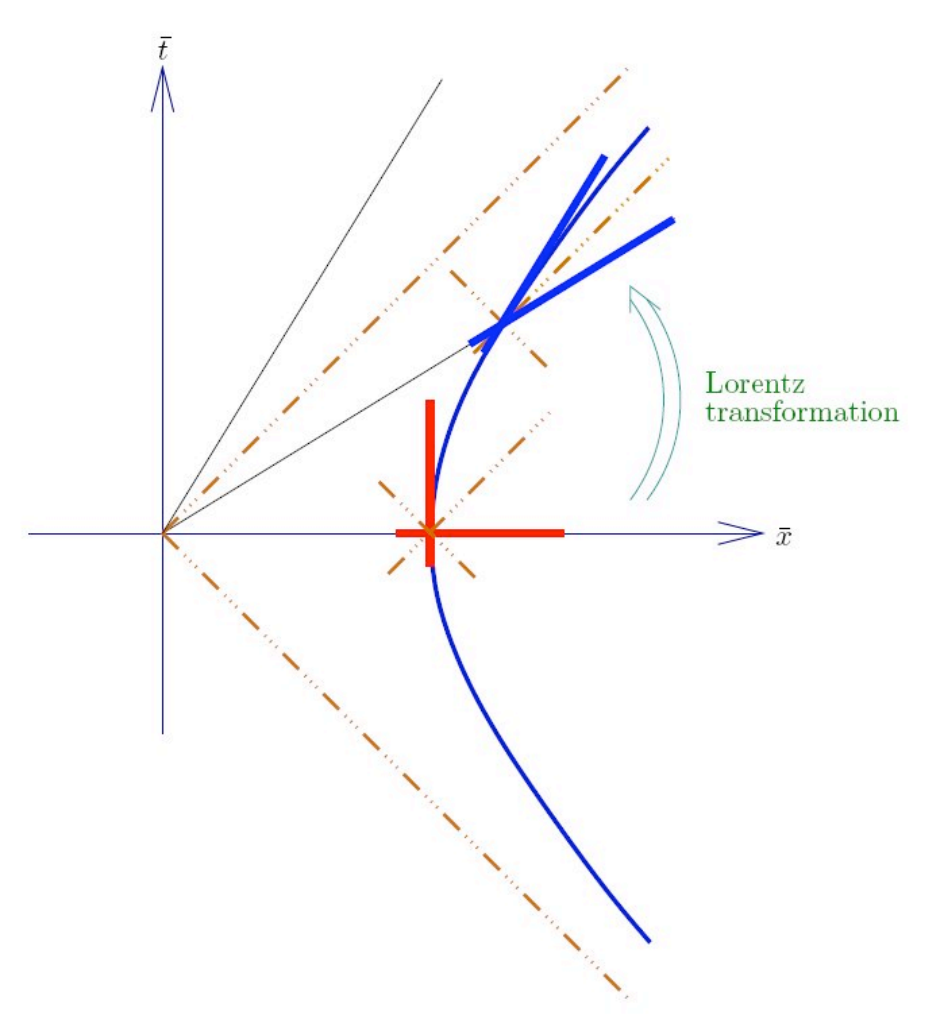
Proof:

We will present a recipe for constructing a world-view for any accelerated observer living on a differentiable "time-like" curve (in a vertical plane):



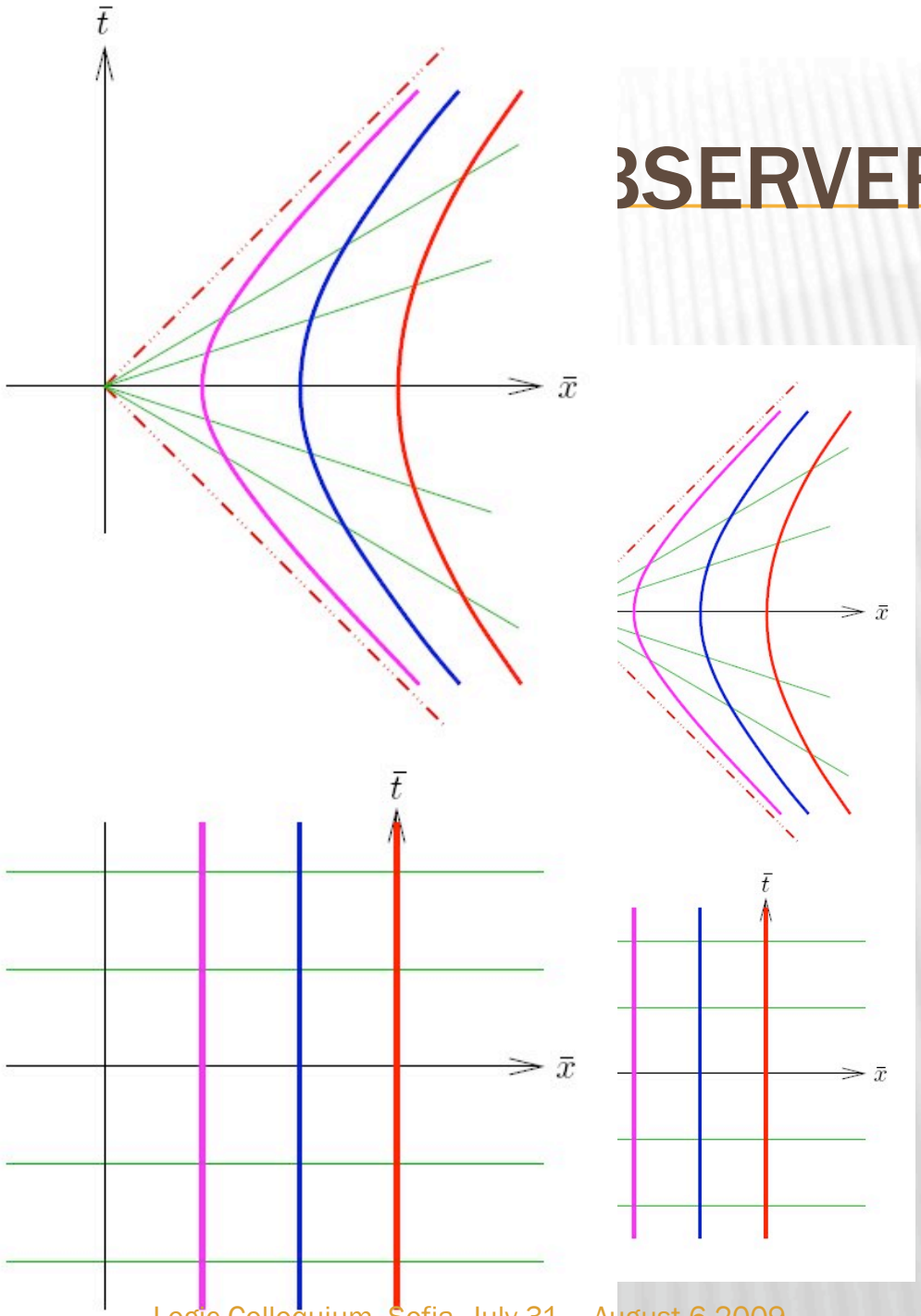
UNIFORMLY ACCELERATED OBSERVERS

World-lines are Minkowski-circles (hyperbolas)

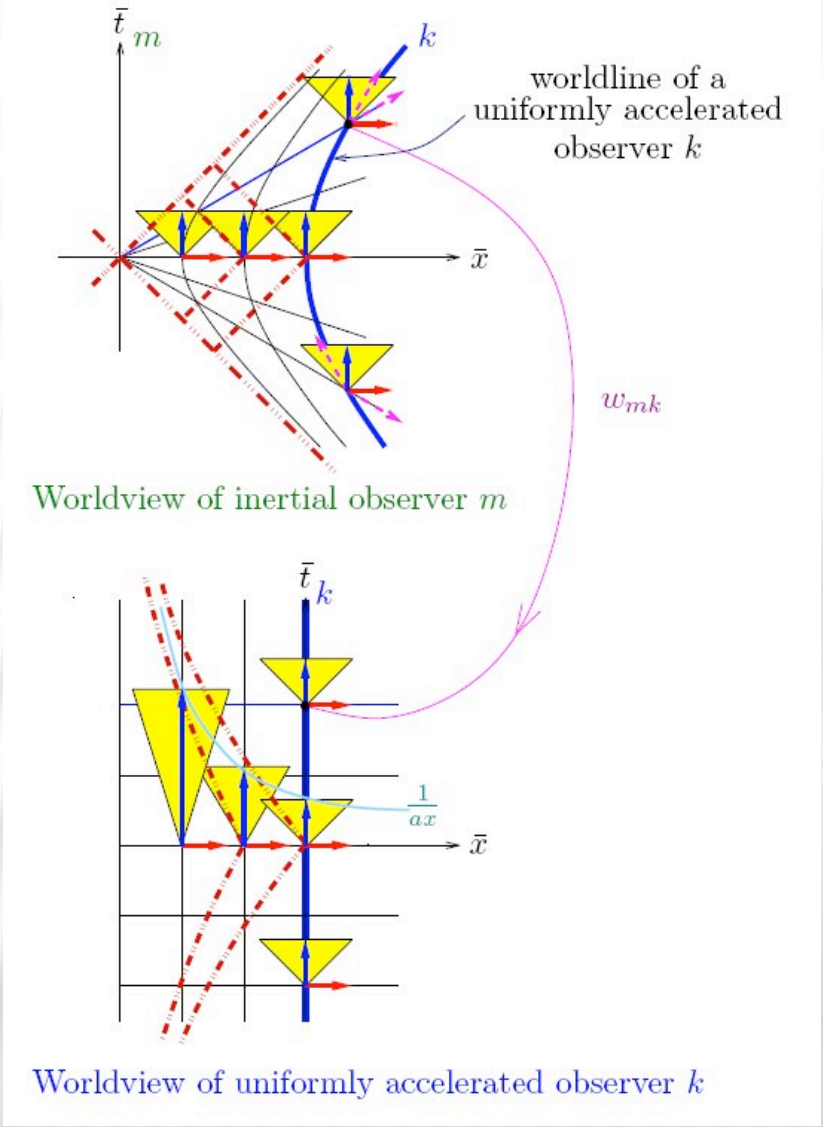


UNIFORMLY A BSERVERS

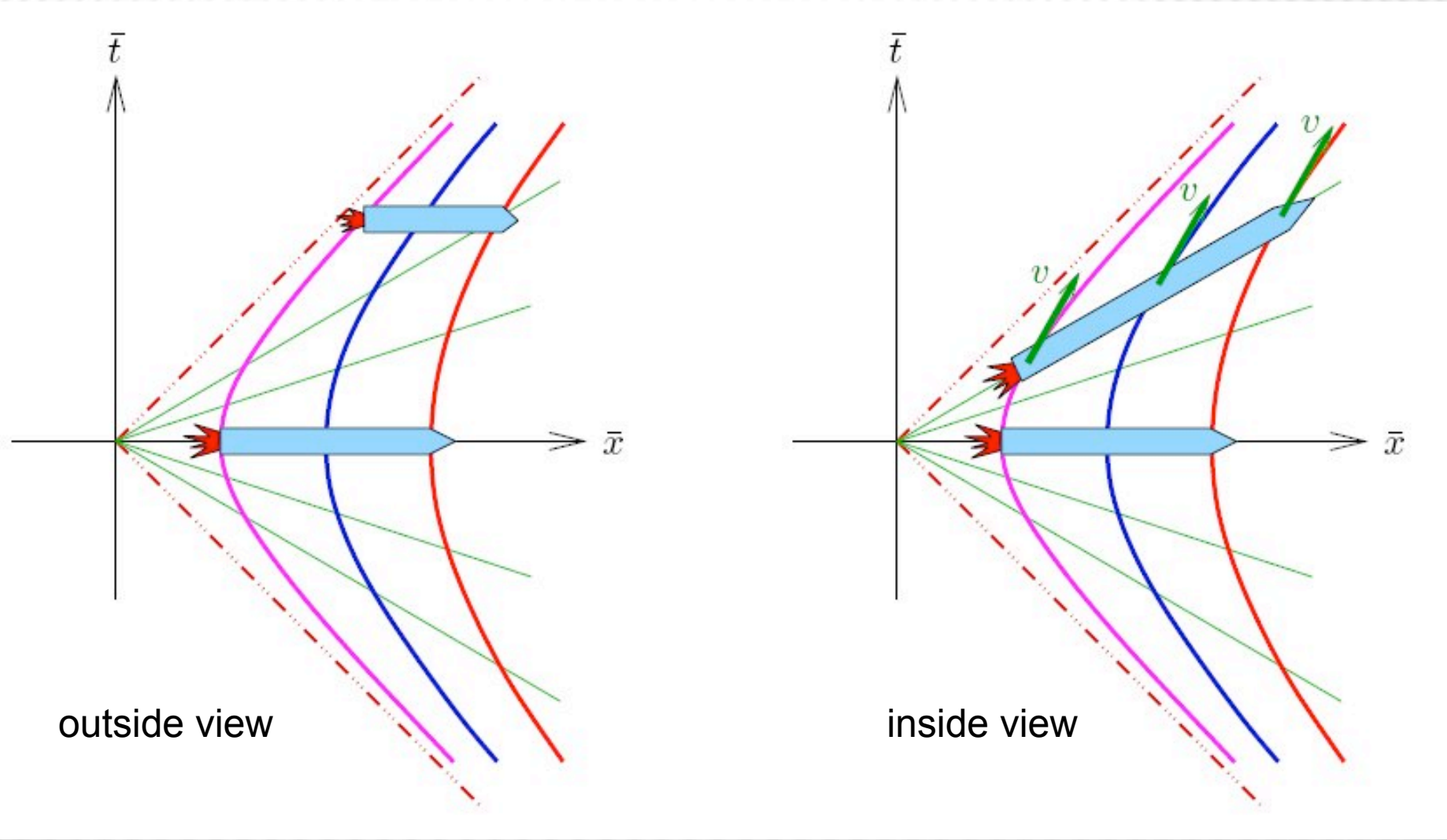
Result of the gener



UNIFORMLY ACCELERATED OBSERVERS

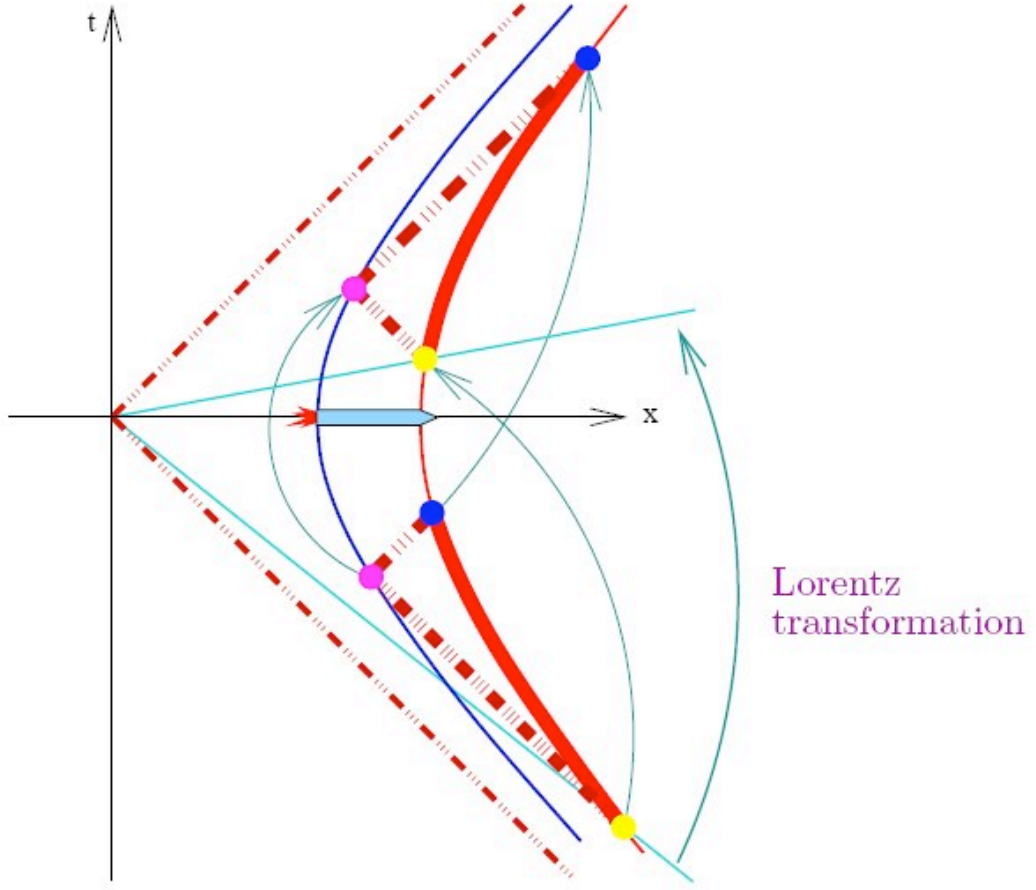


UNIFORMLY ACCELERATING SPACESHIP



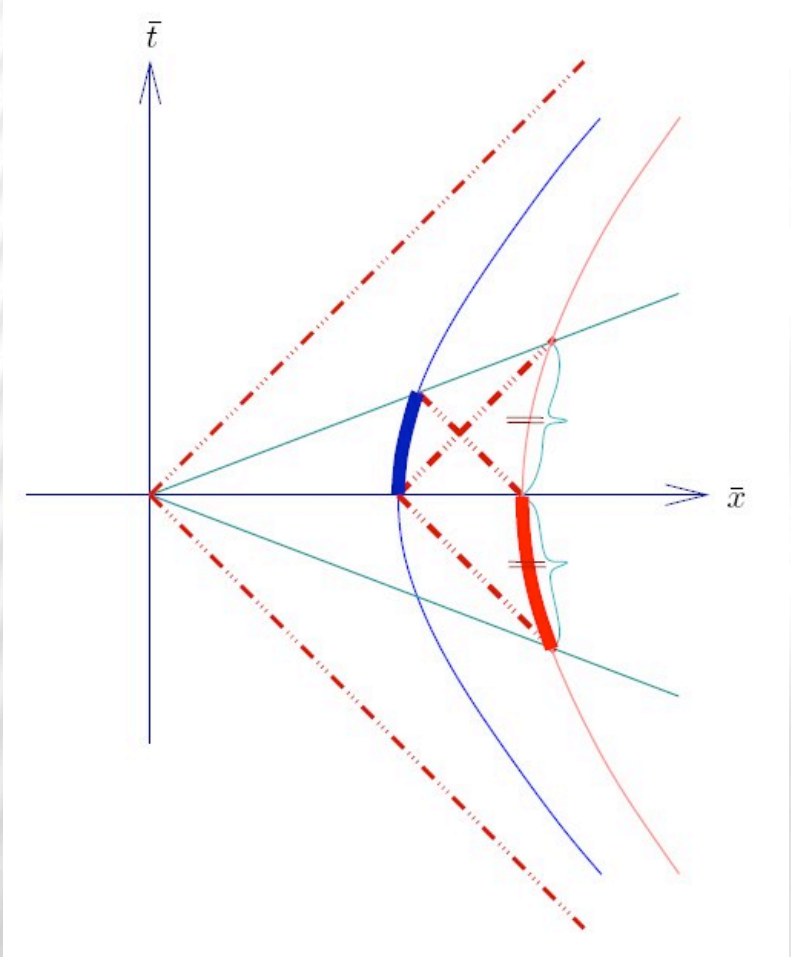
UNIFORMLY ACCELERATING SPACESHIP

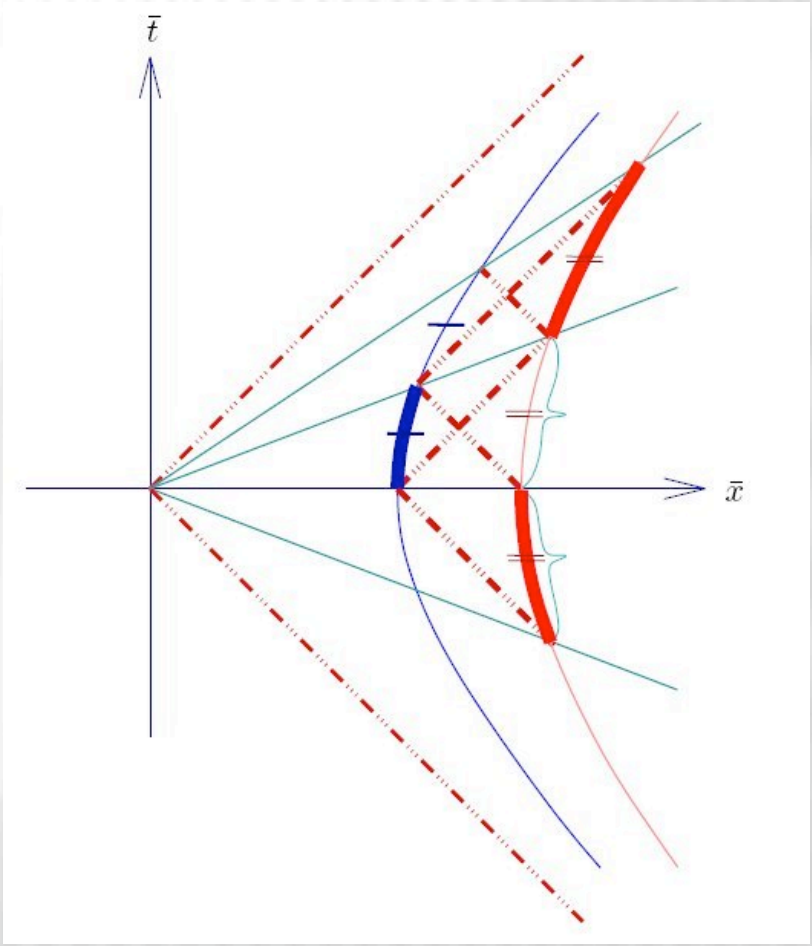
1. Length of the ship does not change seen from inside.
   Captain measures length of his ship by radar:



UNIFORMLY ACCELERATING SPACESHIP

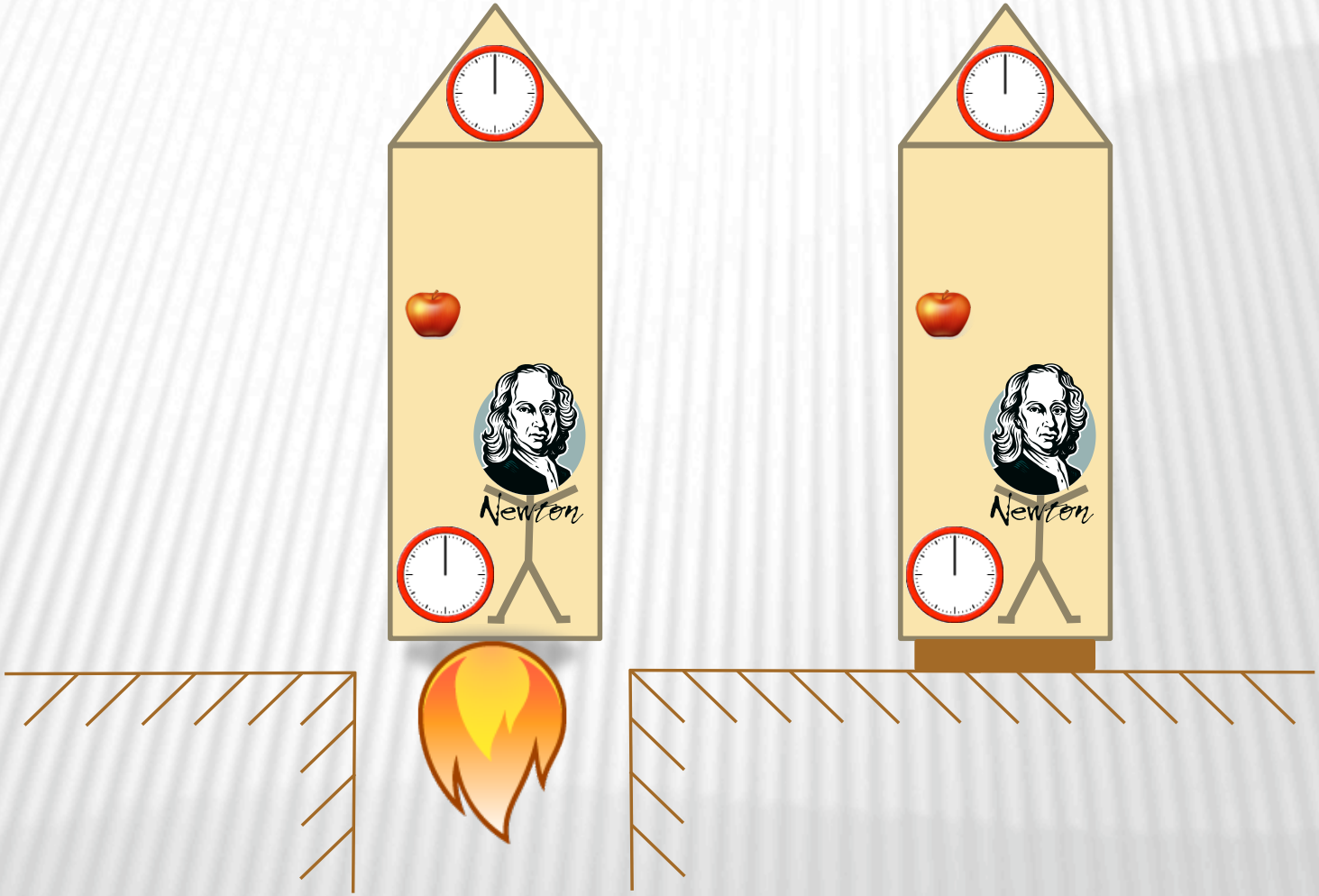
2. Time runs slower at the rear of the ship, and faster at the nose of ship.
Measured by radar:





GRAVITIY CAUSES SLOW TIME

via Einstein's Equivalence Principle



EVERYDAY USE OF RELATIVITY THEORY

GPS

Relativistic
38 μs/day a
-7 μs/day be
ap
+45 μs/day

GRAVITIY CAUSES SLOW TIME

via Einstein's Equivalence Principle

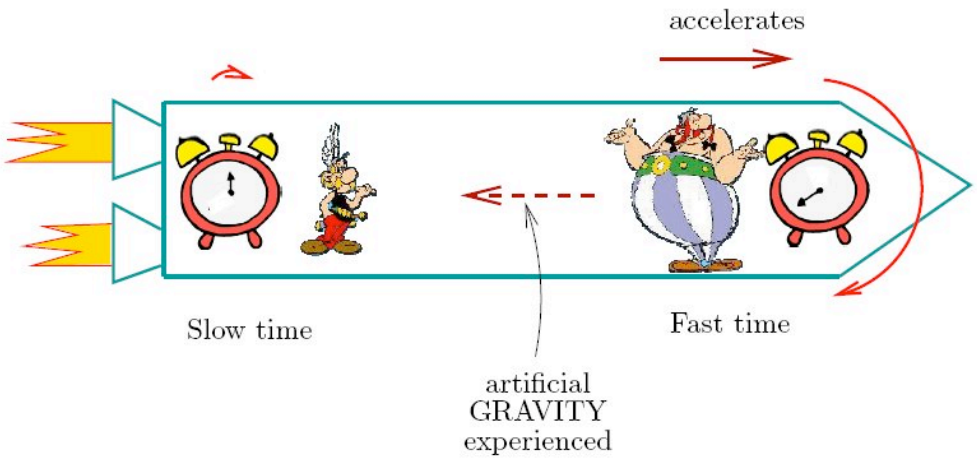


Figure 1: A THEOREM of Special Relativity (SR) (easily proved in first-order logic version of SR).

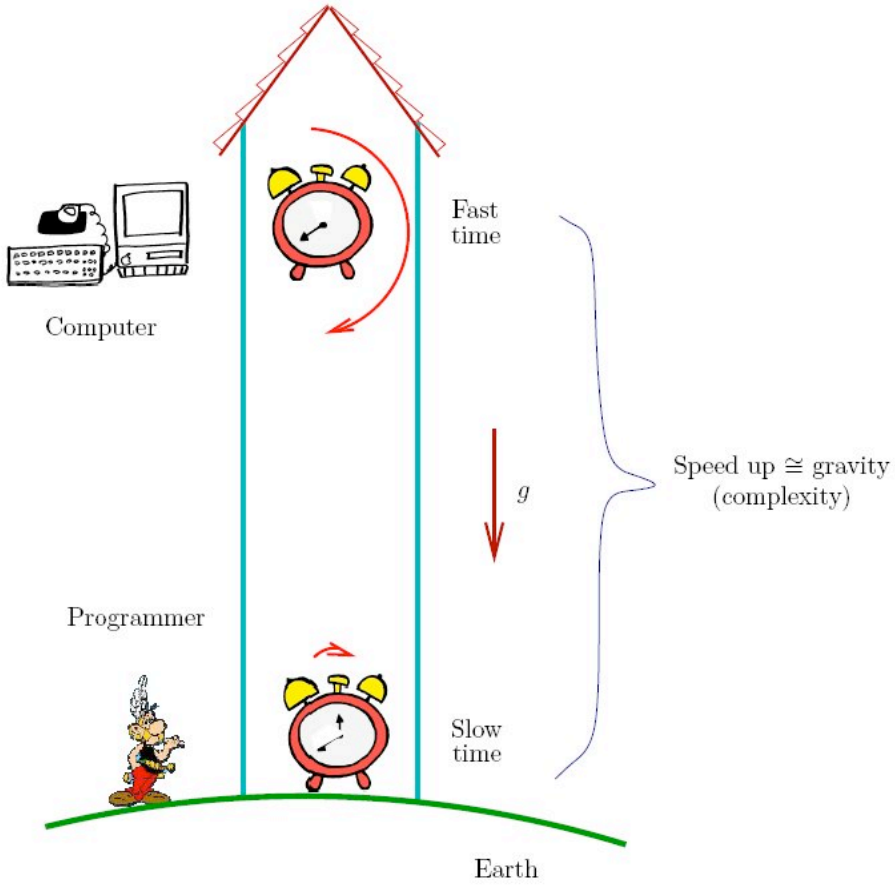


Figure 2: TIME WARP (Tower Paradox, effects of gravity on time). Clocks higher in a gravitational well tick faster.

GRAVITIY CAUSES SLOW TIME

via Einstein's Equivalence Principle

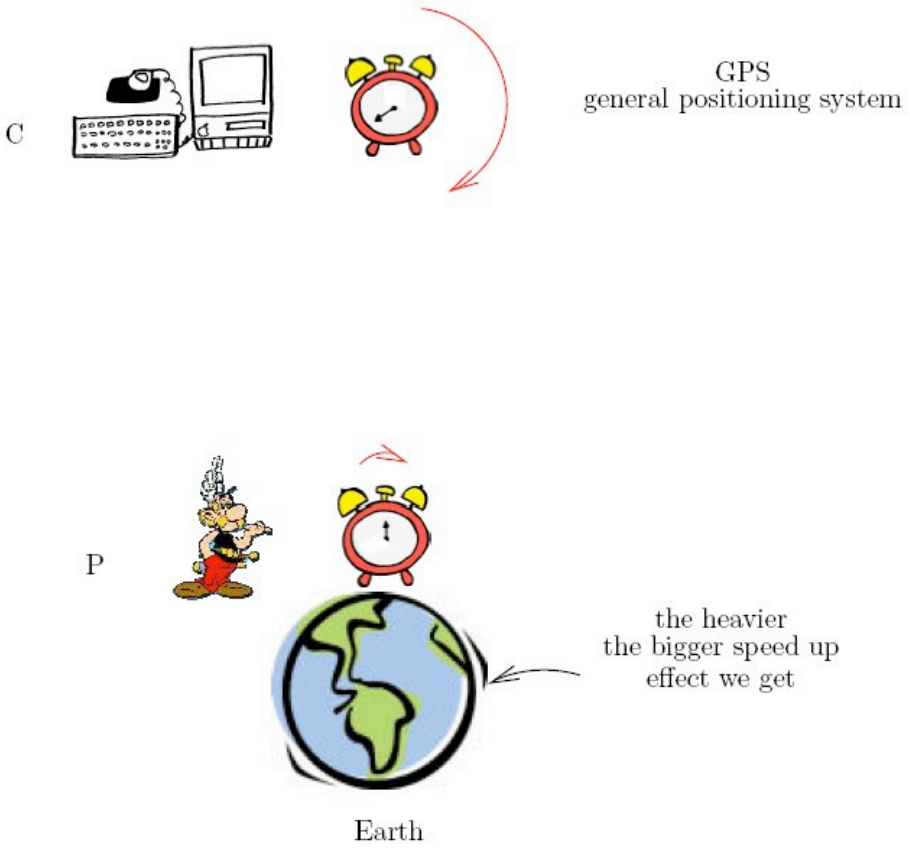


Figure 3: Thought experiment for fast computation: The programmer "throws" his slave-computer to a high orbit. Communicates via radio.

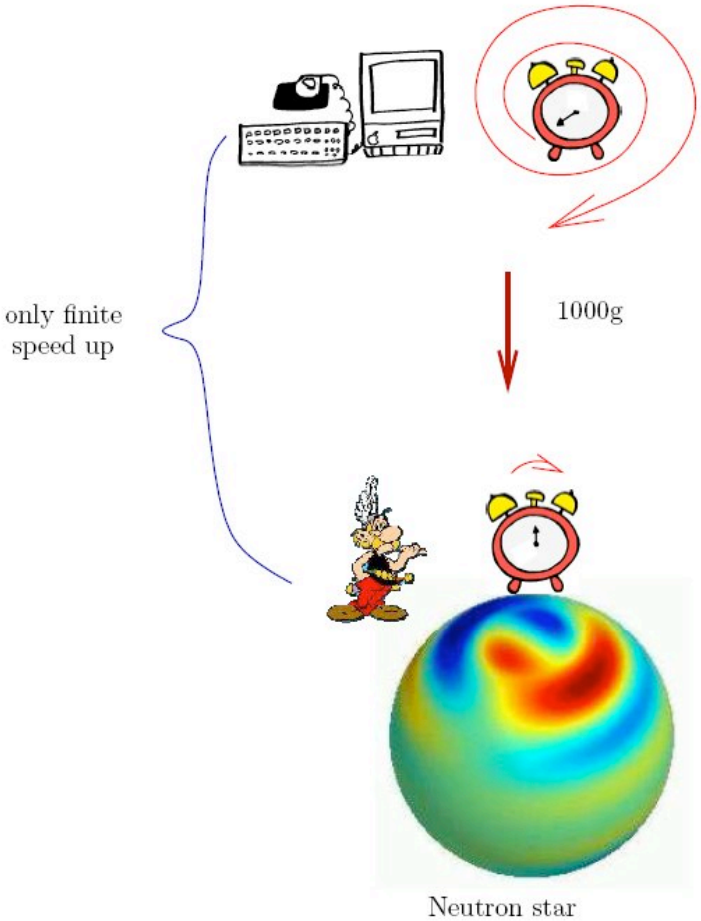


Figure 4: The speed-up effect can be increased by using a neutron star in place of the Earth, but it still remains finite.

GRAVITIY CAUSES SLOW TIME

via Einstein's Equivalence Principle

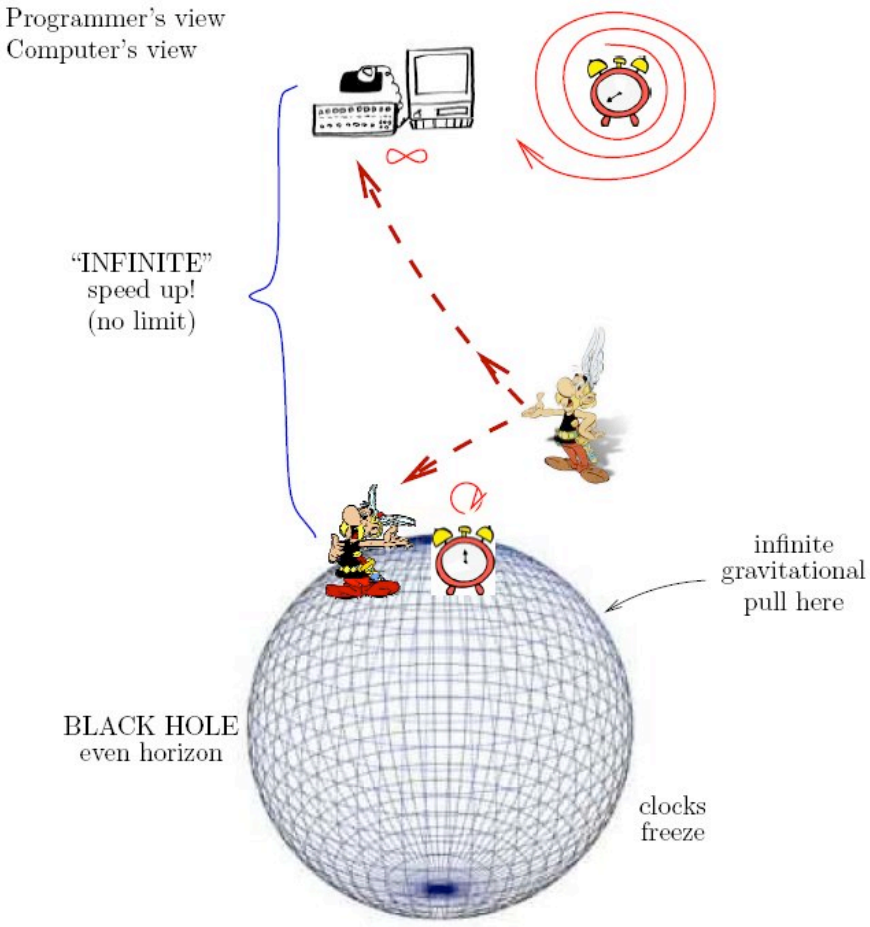


Figure 5: The speed-up effect can be made "infinite" by using a black hole.

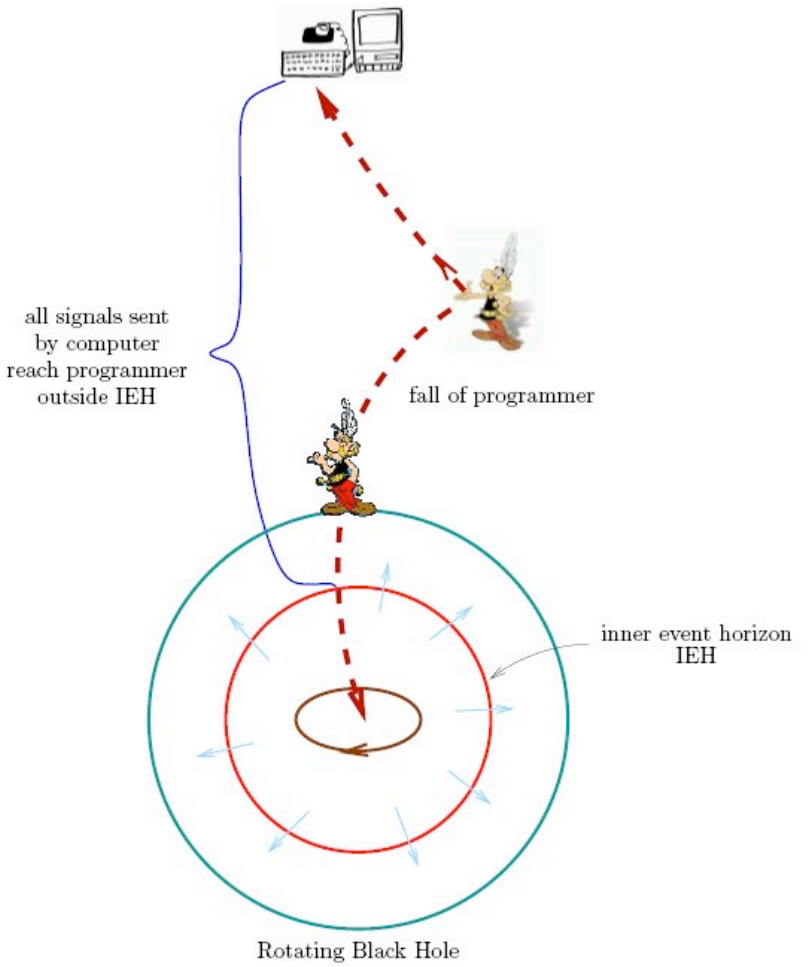
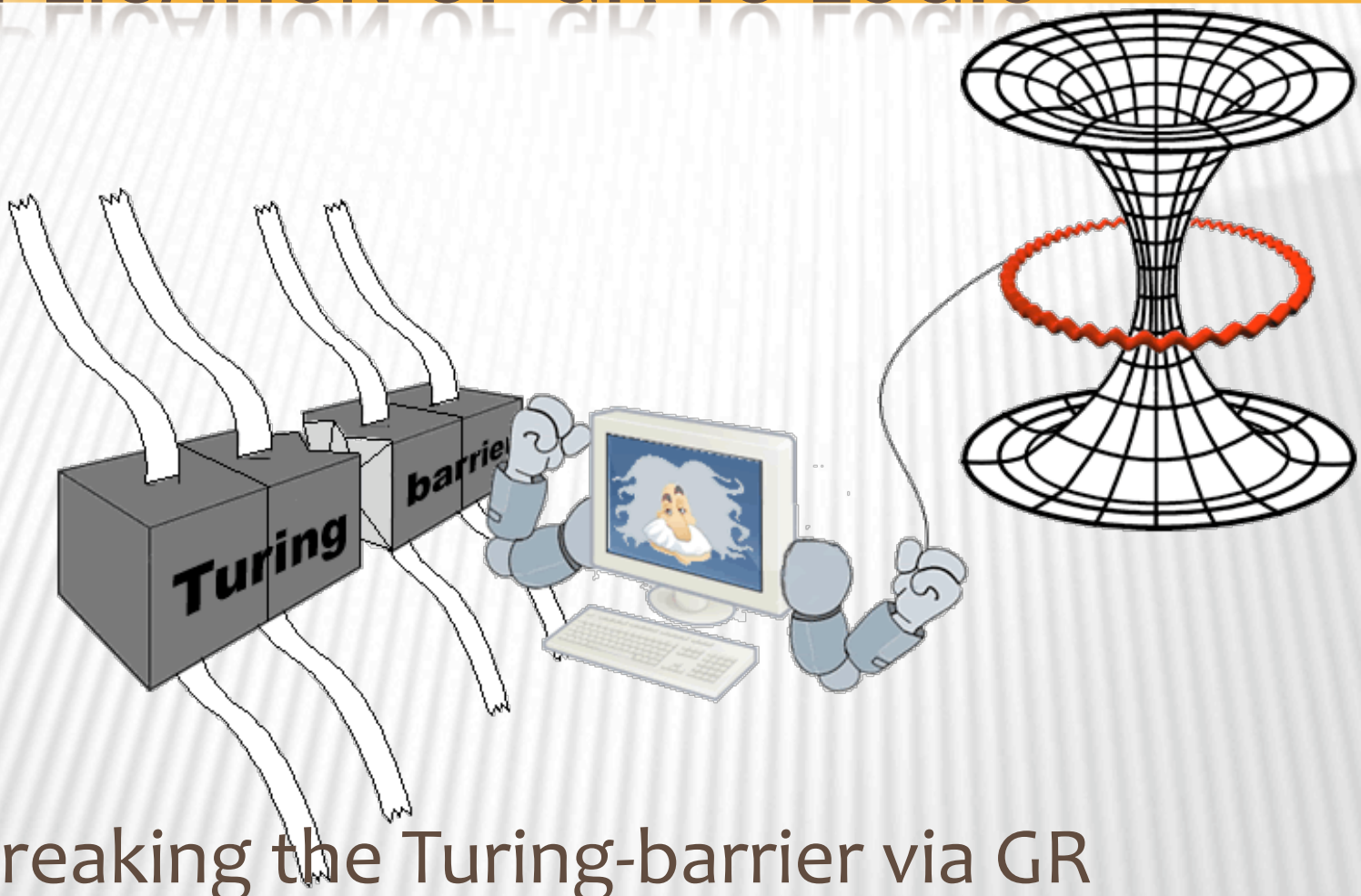


Figure 6: Rotating Black Hole has two event horizons. Programmer can survive forever. (Ring singularity can be avoided.)

APPLICATION OF GR TO LOGIC



Breaking the Turing-barrier via GR

Relativistic Hyper Computing

ACCREL

Conceptual analysis of AccRel goes on …on our homepage

New theory is coming:

GENERAL RELATIVITY

Einstein's strong Principle of Relativity:

"All observers are equivalent" (same laws of nature)

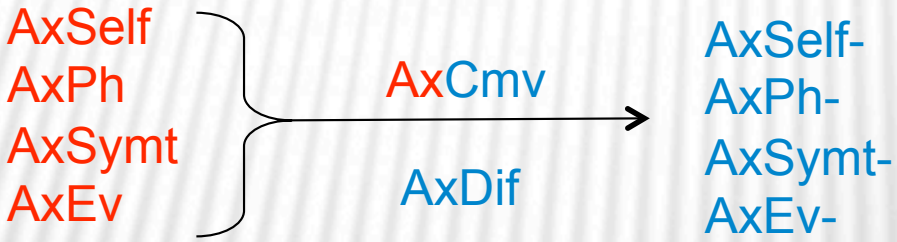
Abolish different treatment of inertial and accelerated observers in the axioms

G. R. ⟹ $GenRel \subseteq FOL$

GENREL

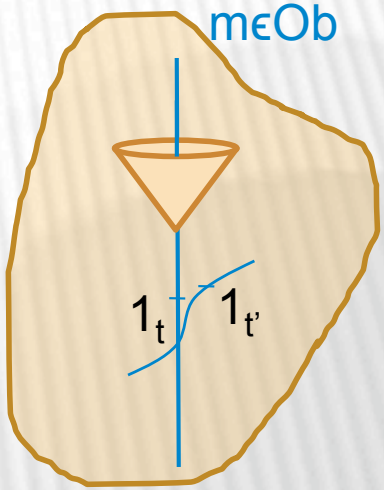
Language of GenRel: the same as that of SpecRel.

Recipe to obtain GenRel from AccRel: delete all axioms from AccRel mentioning IOb. But retain their IOb-free logical consequences.



$$E.g., \quad AxPh + AxCmv \vdash AxPh^{-}$$

AXIOMS FOR GENREL



- AxPh-

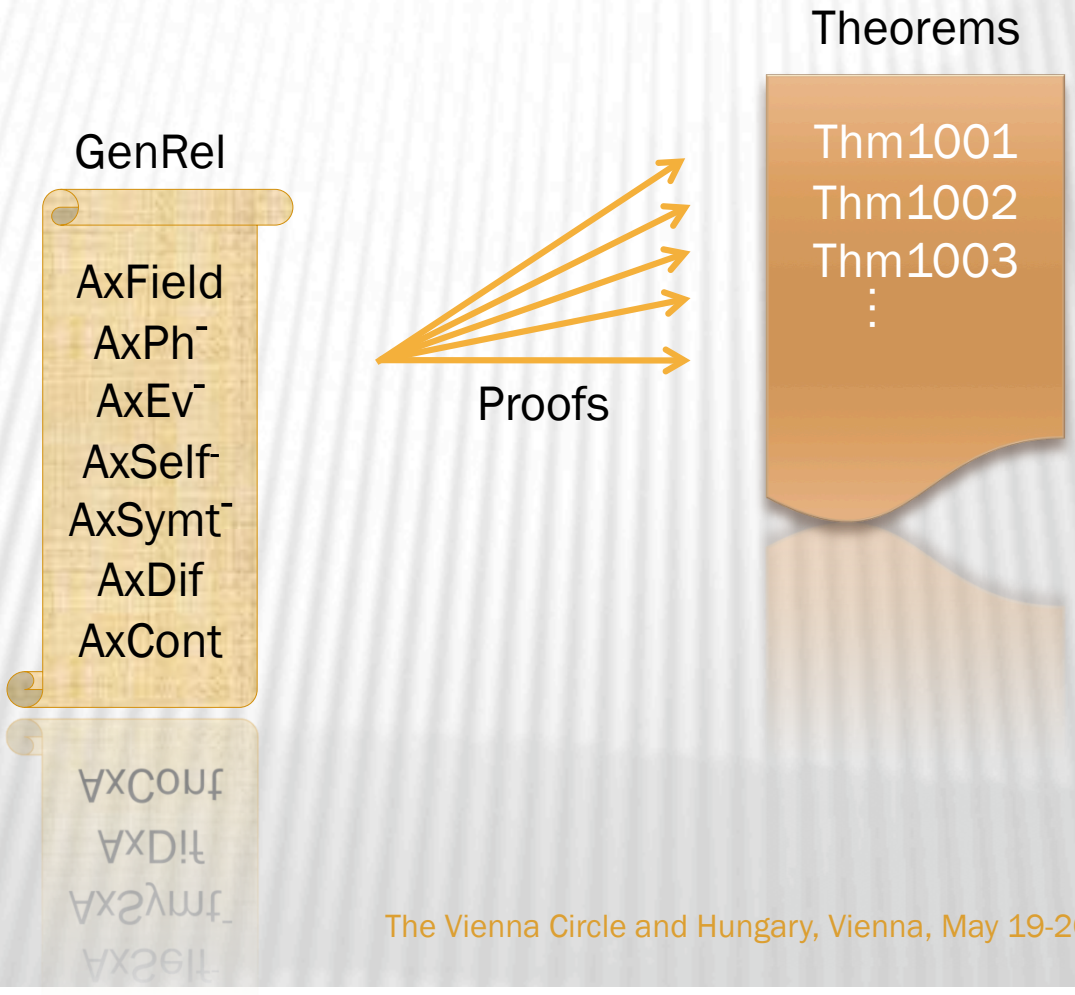
  The velocity of photons an observer "meets" is 1 when they meet, and it is possible to send out a photon in each direction where the observer stands

- AxSym-

  Meeting observers see each other's clocks slow down with the same rate

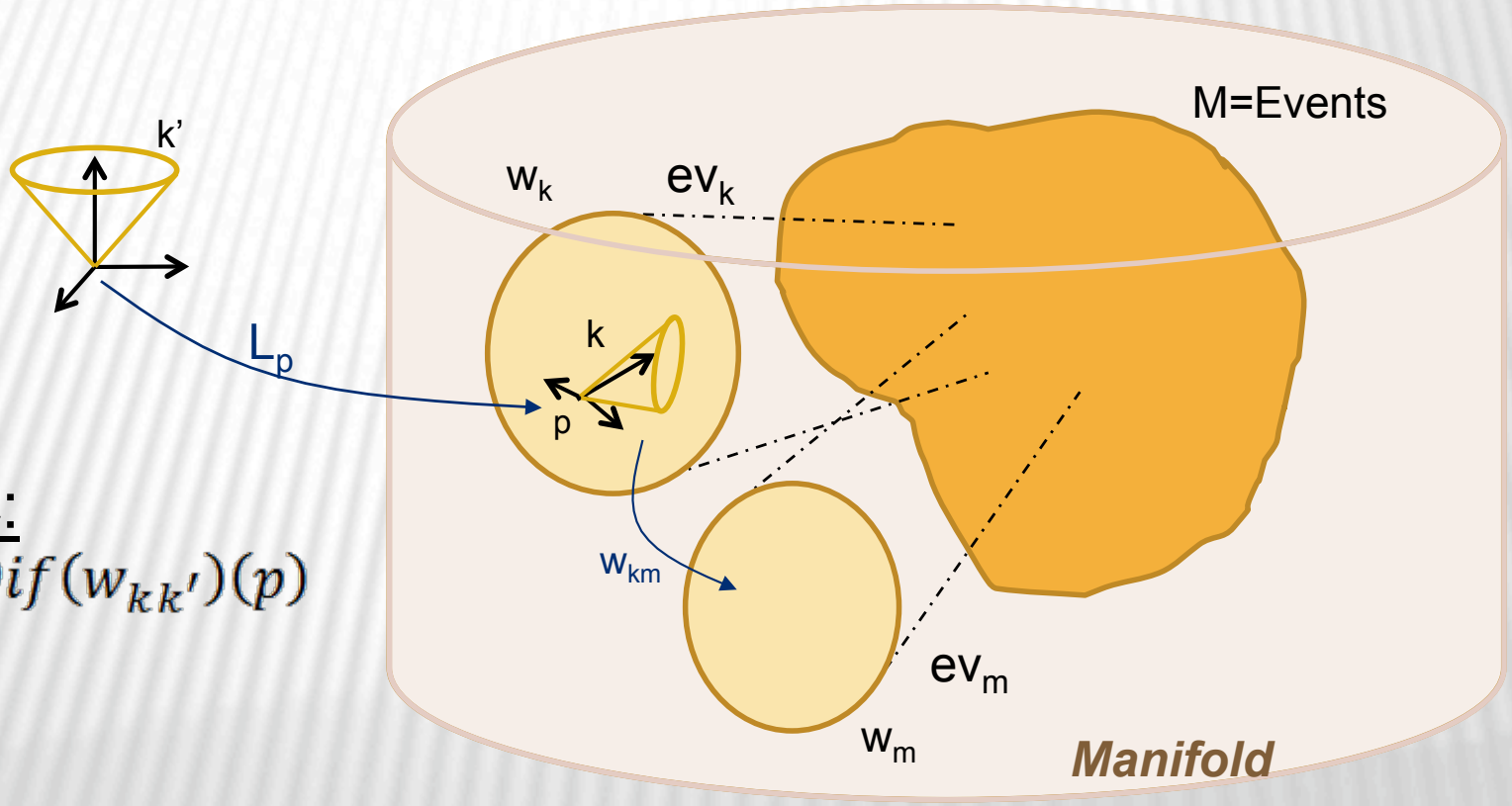
GENREL

$$GenRel = AxField + AxPh^{-} + AxEv^{-} + AxSelf^{-} + AxSymt^{-} + AxDif + AxCont$$



GENREL

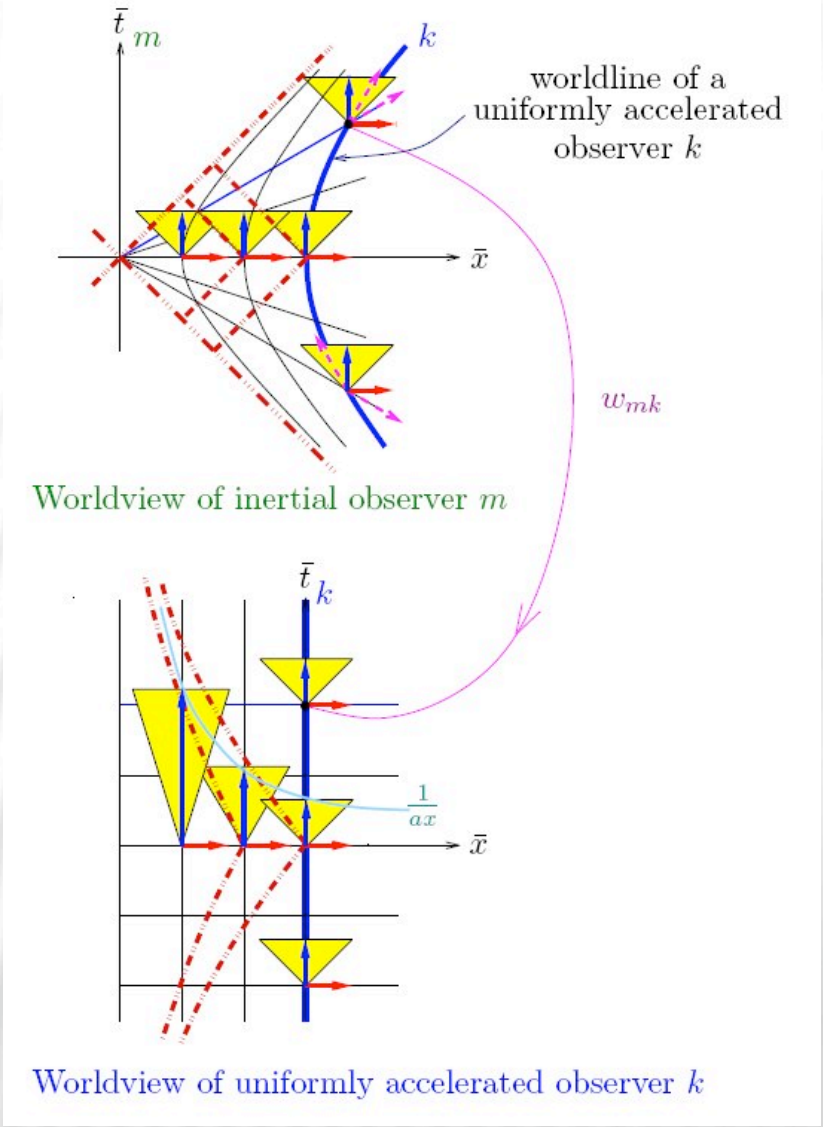
- Thm1002
  - GenRel is complete wrt Lorentz manifolds.



Metric:

$$L_p = Dif(w_{kk'})(p)$$

UNIFORMLY ACCELERATED OBSERVERS



GENREL

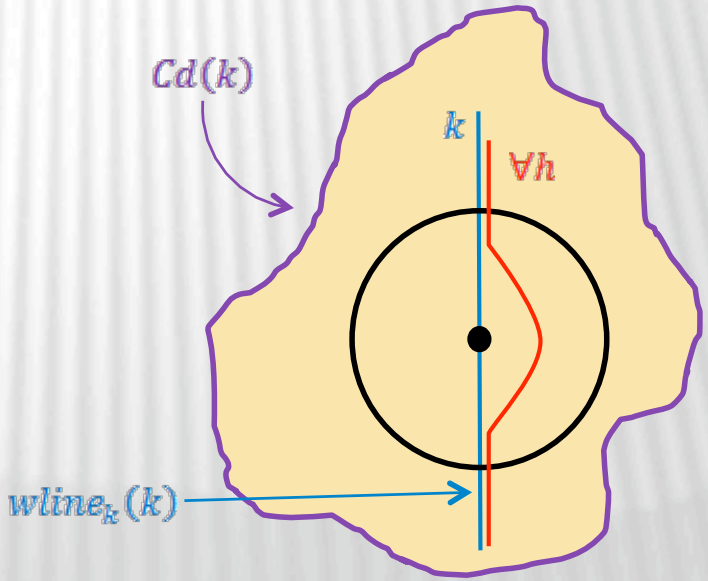
- How to use GenRel?

  Recover IOb's by their properties

  In AccRel by the "twin paradox theorem"

  IOb's are those observers who maximize wristwatch-time locally

  Recover LIFs



GENREL

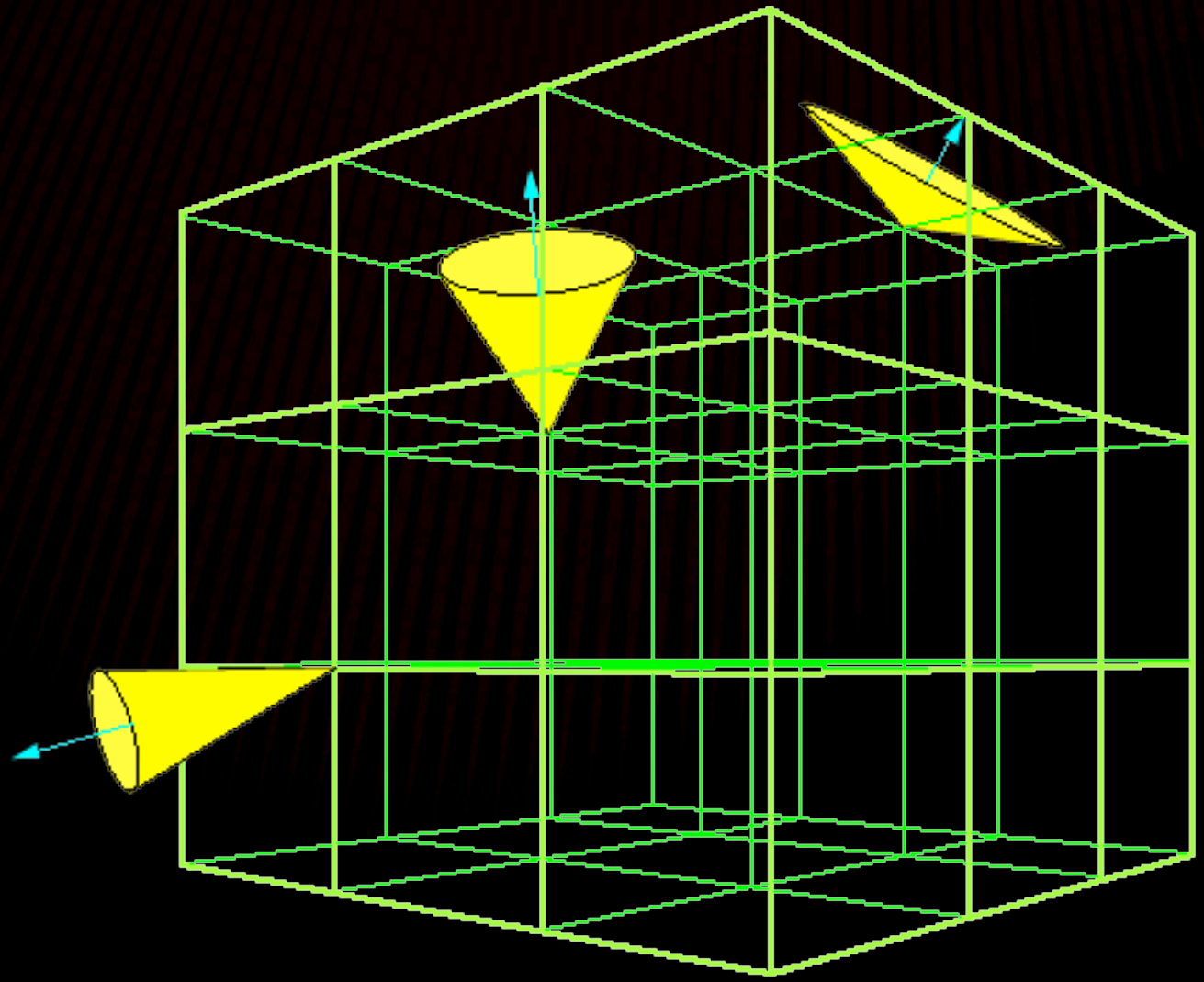
$$Curves \mapsto worldlines\ of\ observers$$

$$AxCompr: Definable\ timelike\ curves\ are\ worldlines\ of\ possible\ observers$$

Timelike means: exist comoving observer

$$GenRel^{+} := GenRel + AxCompr$$

SPECIFYING A GENREL SPACE-TIME



EXAMPLES FOR GENREL SPACETIMES

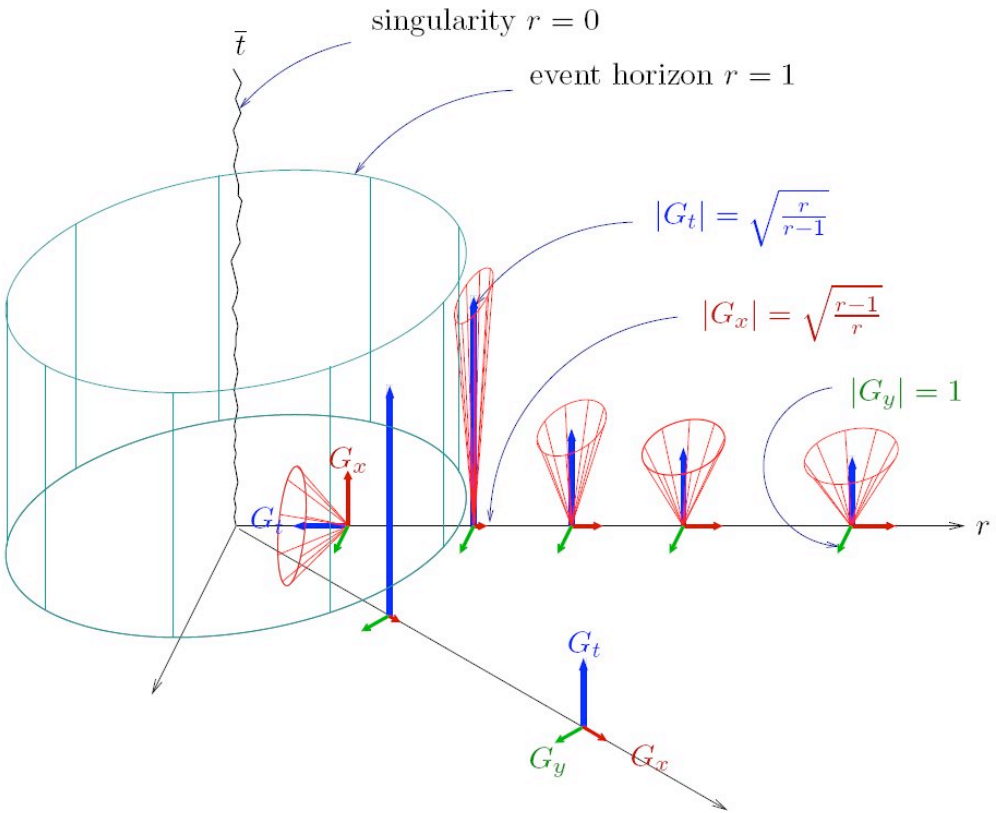
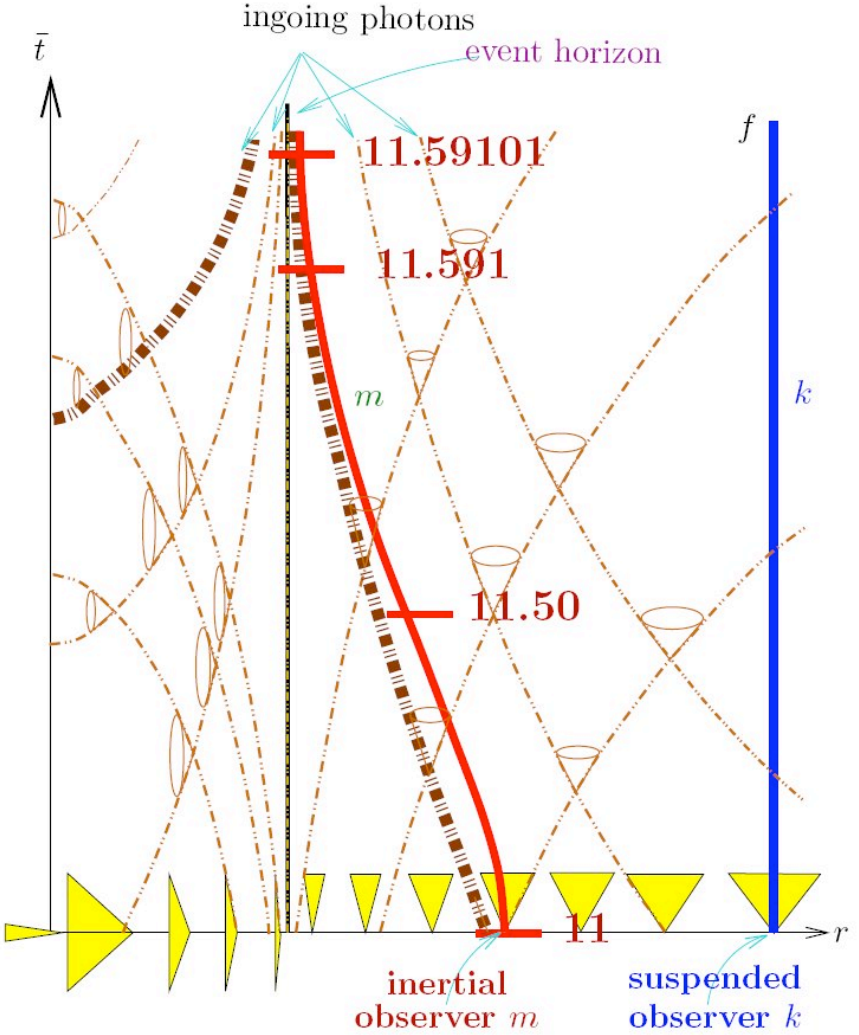


Figure 0.7. Spacetime of Schwarzschild black hole.



EXAMPLES FOR GENREL SPACETIMES

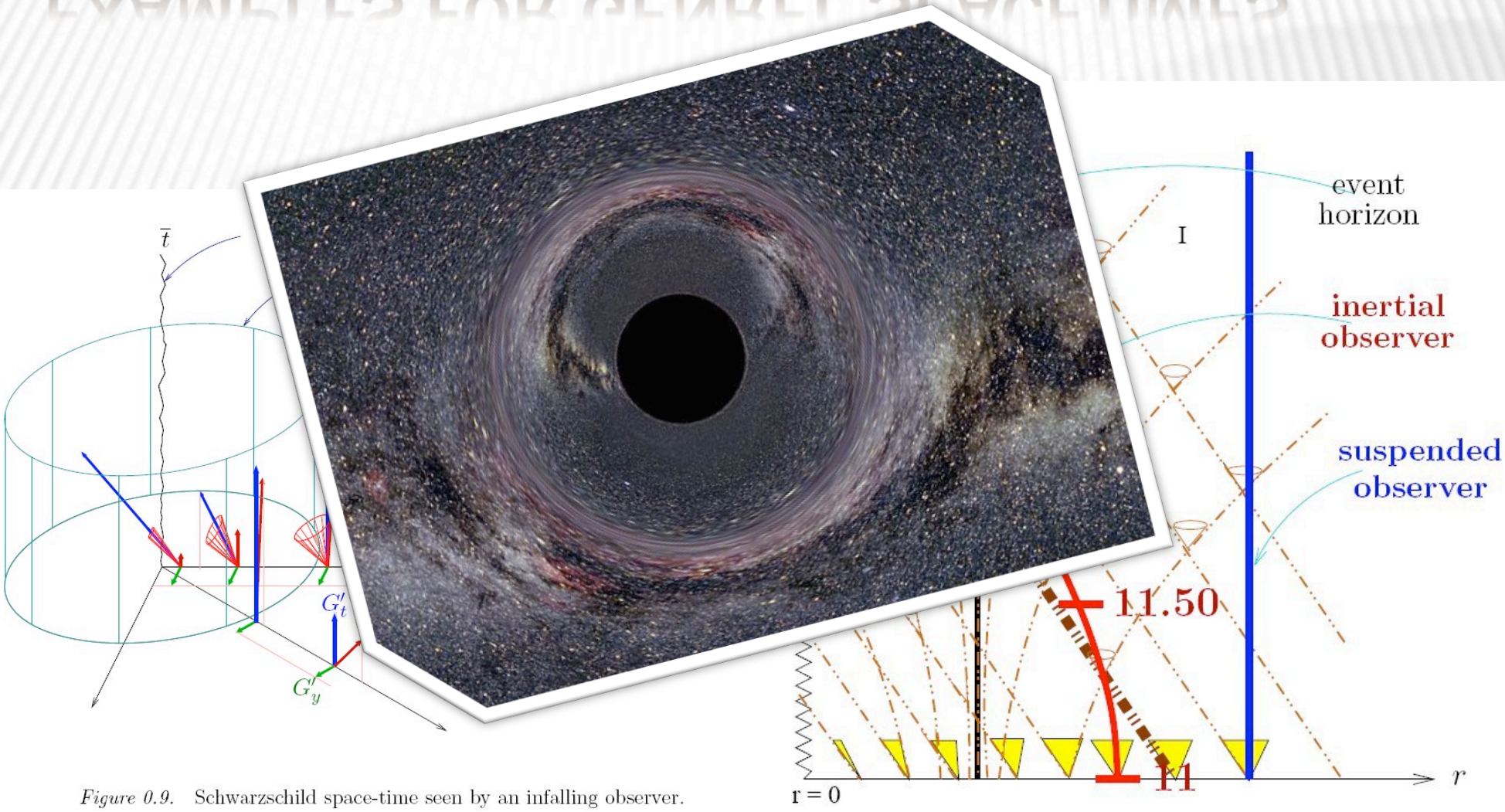


Figure 0.9. Schwarzschild space-time seen by an infalling observer.

EXAMPLES FOR GENREL SPACETIMES

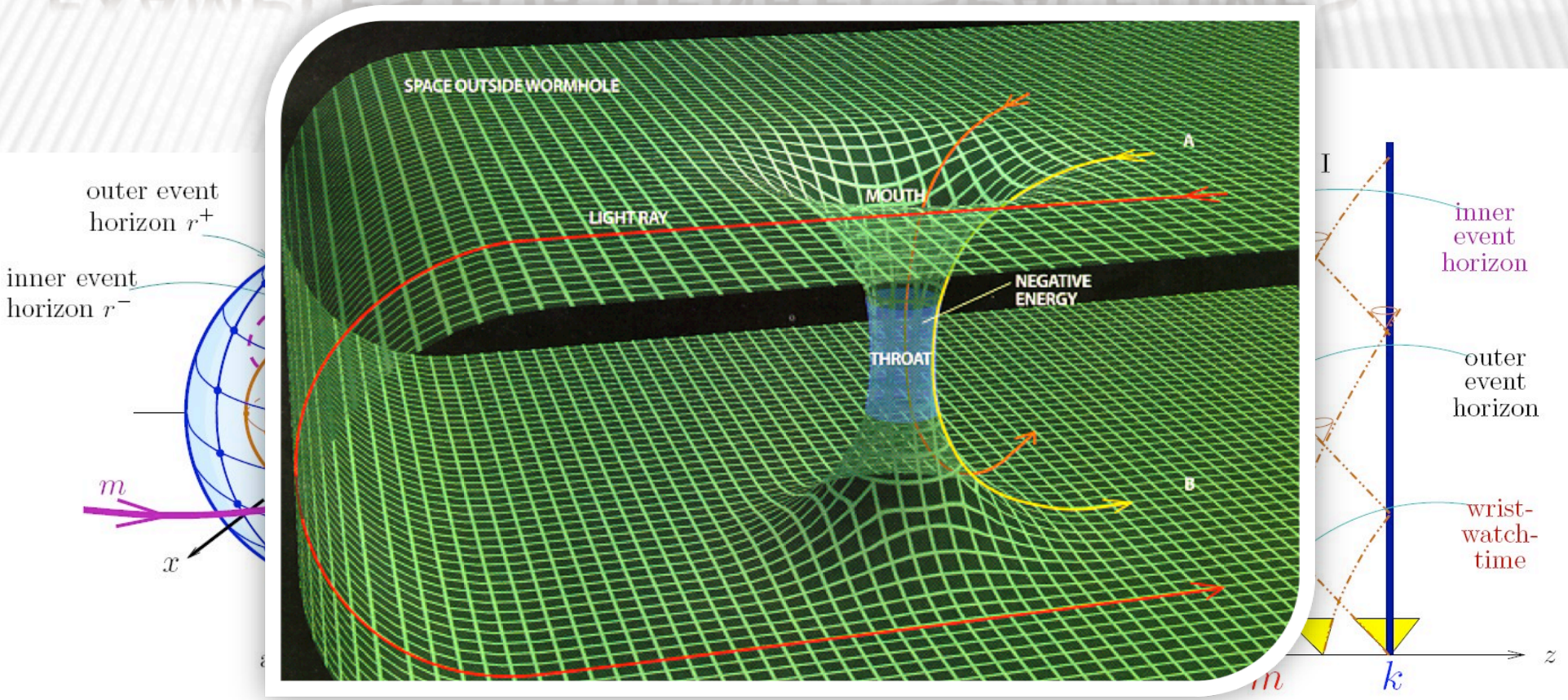


Figure 0.11. Slowly rotating (Kerr) black hole.

Figure 0.12. The "tz-slice" of space-time of slowly rotating black hole.

EXAMPLES FOR GENREL SPACETIMES

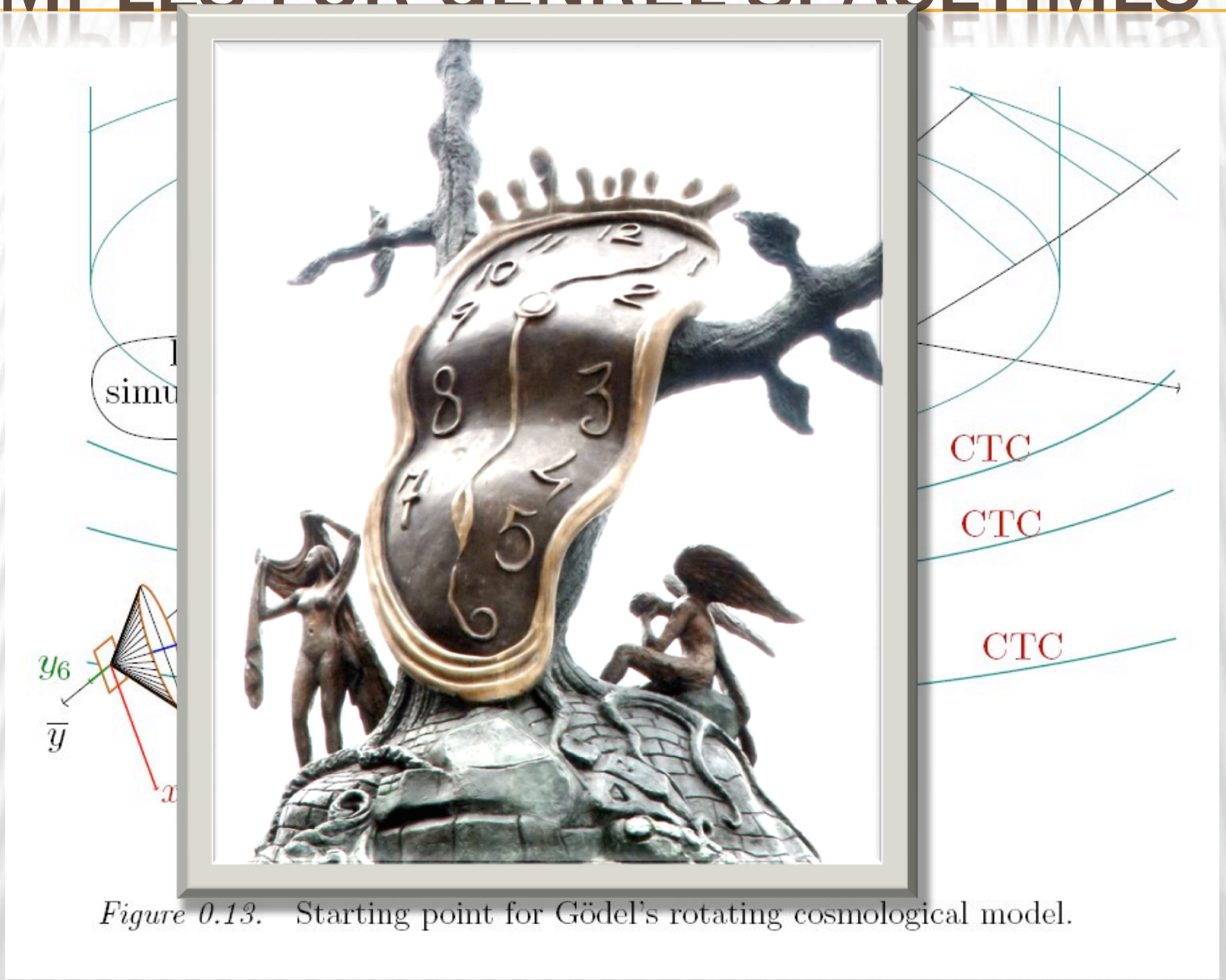
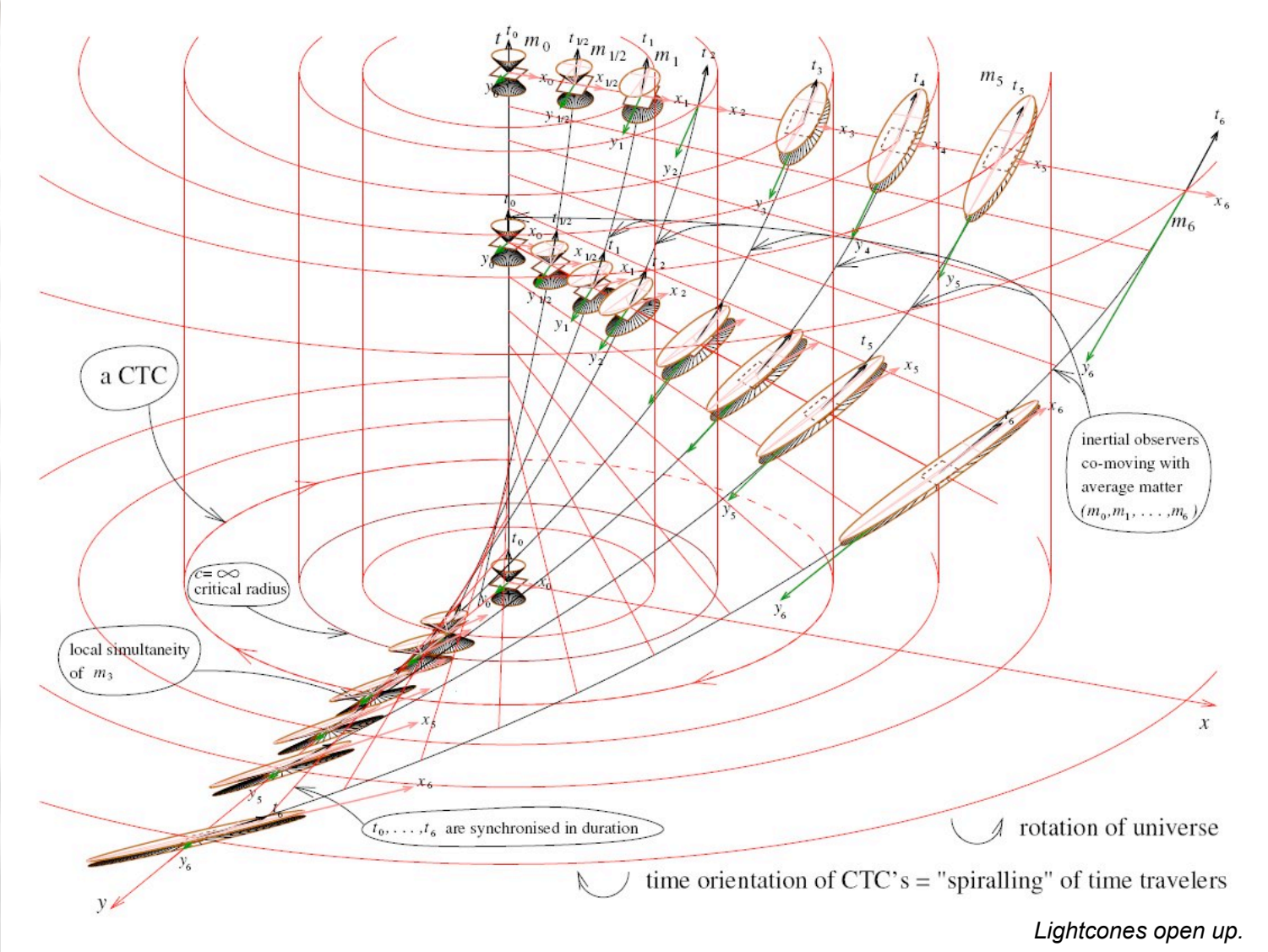


Figure 0.13. Starting point for Gödel's rotating cosmological model.

EXAMPLES FOR GENREL SPACETIMES



Lightcones open up.

More in our papers in **General Relativity and Gravitation** 2008 & in **arXiv.org** 2008

PUBLICATIONS

More concrete material available from our group:

(1) Logic of Spacetime
http://ftp.math-inst.hu/pub/algebraic-logic/Logicofspacetime.pdf

(2) in Foundation of Physics
http://ftp.math-inst.hu/pub/algebraic-logic/twp.pdf

(3) First-order logic foundation of relativity theories
http://ftp.math-inst.hu/pub/algebraic-logic/springer.2006-04-10.pdf

(4) FOL 75 papers
http://www.math-inst.hu/pub/algebraic-logic/foundrel03nov.html
http://www.math-inst.hu/pub/algebraic-logic/loc-mnt04.html

(5) our e-book on conceptual analysis of SpecRel
http://www.math-inst.hu/pub/algebraic-logic/olsort.html

(6) More on István Németi's homepage http://www.renyi.hu/~nemeti/
(Some papers available, and some recent work)

Thank you!