Complete axiomatizations of modal logics for region-based theories of space

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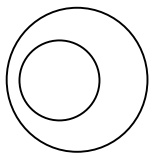
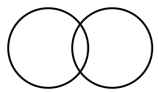
Introduction

Introduction

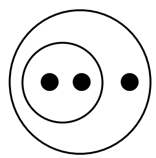
- Region-based theory of space
  - Spatial entities
    - Regions
  - Spatial relations
    - Part-of
    - Contact (connection)

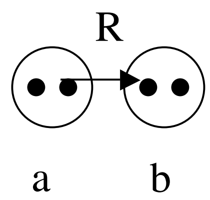




Introduction

- Adjacency spaces
  - (W,R)
  - Spatial entities
    - Regions: sets of cells
  - Spatial relations
    - Part-of: inclusion
    - Contact: a and b are in contact iff for some x∈W and y∈W we have x∈a, xRy and y∈b





Introduction

- Modal logics for region-based theories of space
  - Boolean variables: p_1, p_2, …
  - Boolean operations: 0, *, ∪
  - Boolean terms
    - $a ::= 0 \mid a^* \mid (a \cup b)$
  - Modal connectives: ≤ (part-of), C (contact)
  - Propositional connectives: ⊥, ¬, ∨
  - Modal formulas
    - $\phi ::= (a \leq b) \mid (aCb) \mid \bot \mid \neg\phi \mid (\phi \vee \psi)$

Introduction

- Outline
  - Syntax and relational semantics
  - Modal definability and undefinability
  - Axiomatizations and completeness
  - Filtration and small canonical models
  - Logics related to the colourability of graphs
  - Logics related to RCC
  - Extensions with rules of inference
  - Some complexity results
  - Topological models

Syntax and relational semantics

Syntax and relational semantics

- Syntax
  - Language
    - Boolean variables: p_1, p_2, …
    - Boolean operations: 0, *, $\cup$
    - Modal connectives: ≤ (part-of), C (contact)
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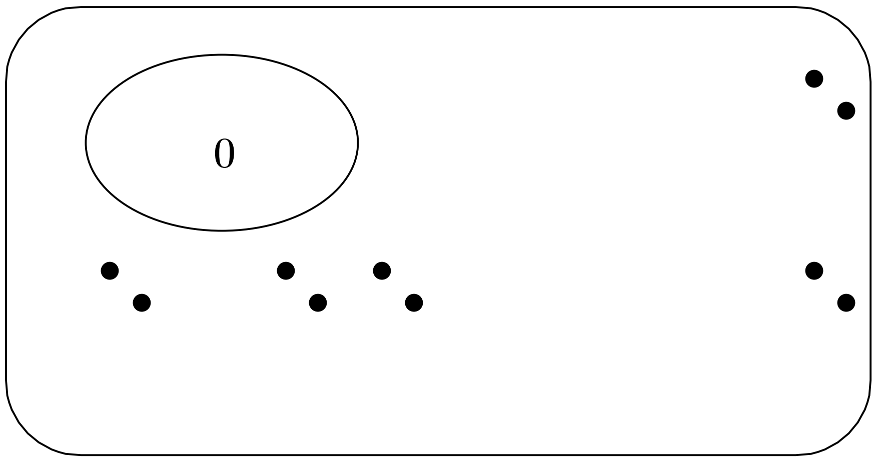

Syntax and relational semantics

- Syntax
  - Language
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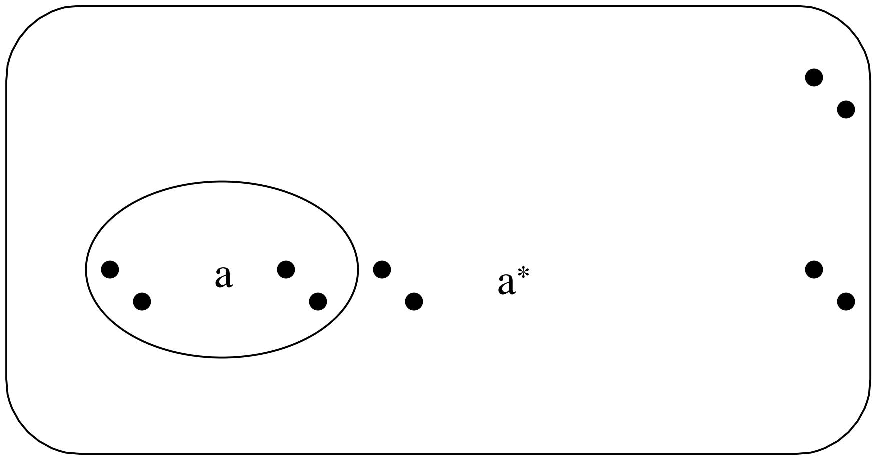

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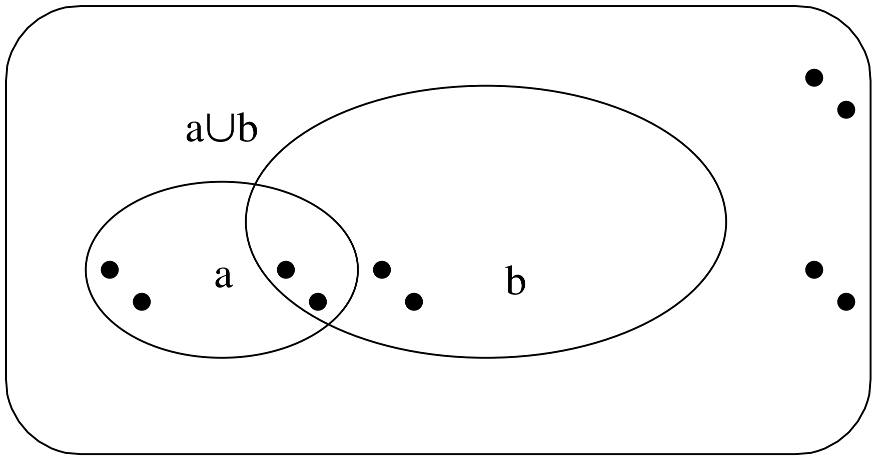

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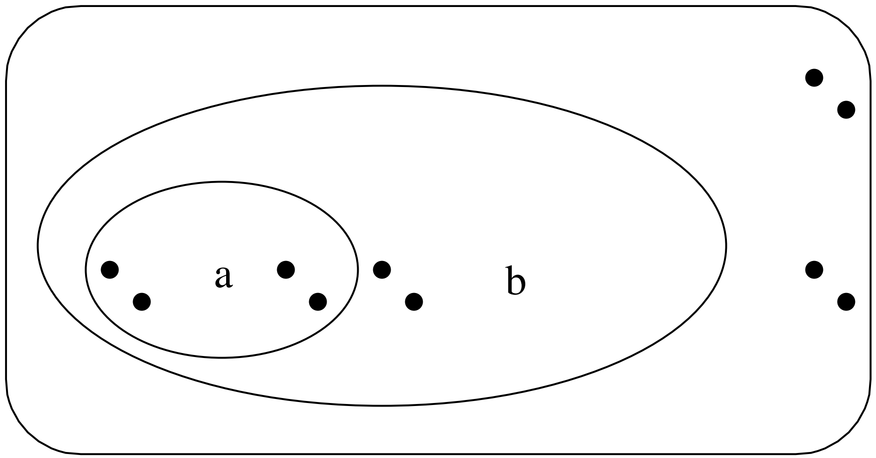
(a≤b)



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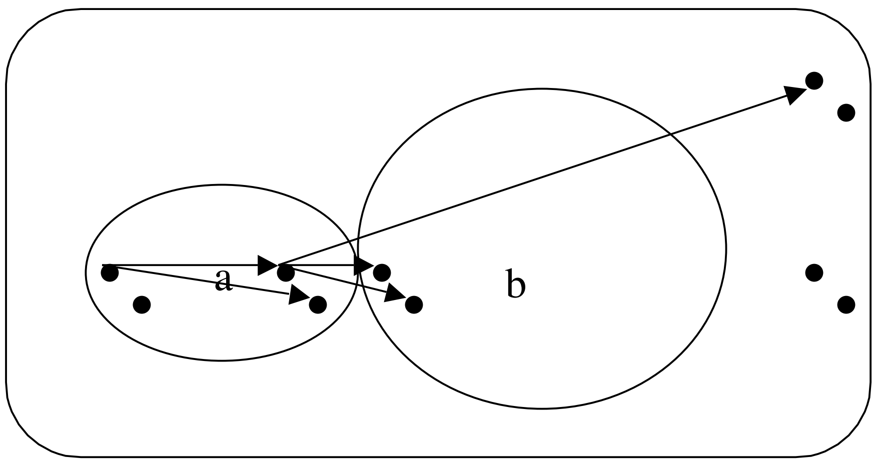
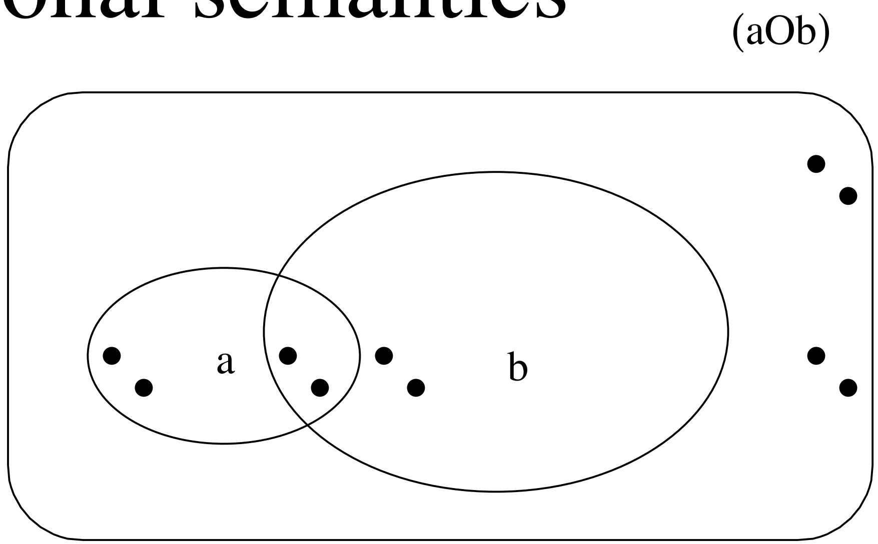
(aCb)



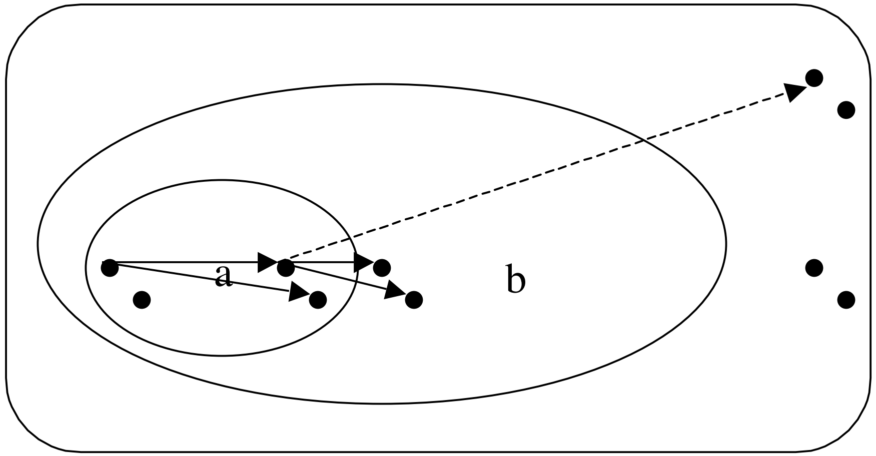
Syntax and relational semantics



- Syntax
  - Abbreviations
    - (a=b) ::= (a≤b)∧(b≤a)
    - (a≠b) ::= ¬(a=b)
    - <u>(aOb)</u> ::= (a∩b≠0) (overlap)
    - (a<<b) ::= ¬(aCb*) (non-tangential inclusion)
  - Substitution
    - $a(p_1,\ldots,p_n) / a(a_1,\ldots,a_n)$, $\phi(p_1,\ldots,p_n) / \phi(a_1,\ldots,a_n)$
    - $\varphi(x_1,\ldots,x_n) / \varphi(\phi_1,\ldots,\phi_n)$

Syntax and relational semantics

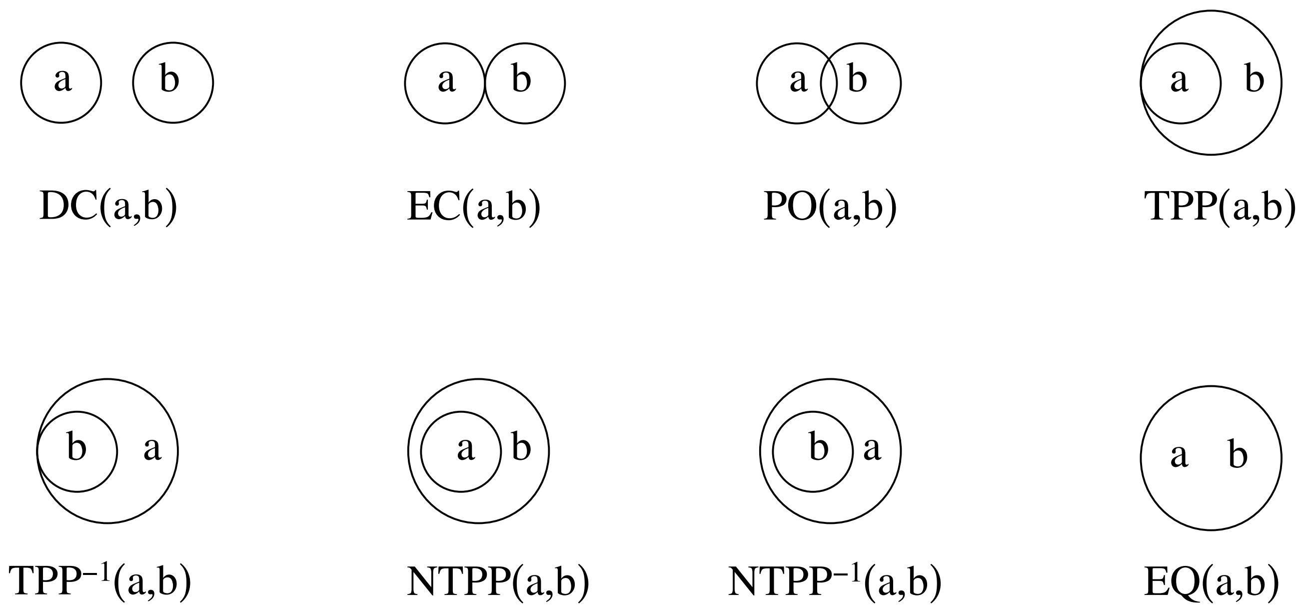
(a<<b)



- Syntax
  - Abbreviations
    - $(a=b) ::= (a \leq b) \wedge (b \leq a)$
    - $(a \neq b) ::= \neg(a=b)$
    - $(aOb) ::= (a \cap b \neq 0)$ (overlap)
    - <u>(a<<b)</u> $::= \neg(aCb^*)$ (non-tangential inclusion)
  - Substitution
    - $a(p_1,\ldots,p_n)/a(a_1,\ldots,a_n)$, $\phi(p_1,\ldots,p_n)/\phi(a_1,\ldots,a_n)$
    - $\varphi(x_1,\ldots,x_n)/\varphi(\phi_1,\ldots,\phi_n)$

Syntax and relational semantics

- RCC-8 relations

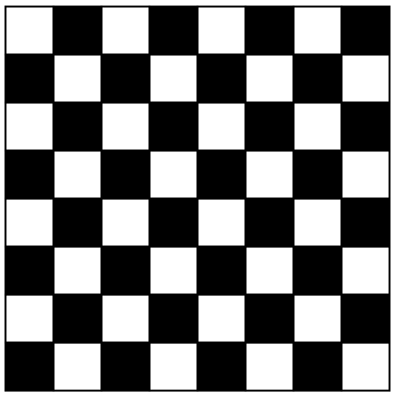


Syntax and relational semantics

- RCC-8 relations
  - Disconnected: $DC(a,b) ::= \neg(aCb)$
  - External contact: $EC(a,b) ::= (aCb) \wedge \neg(aOb)$
  - Partial overlap: $PO(a,b) ::= (aOb) \wedge \neg(a \leq b) \wedge \neg(b \leq a)$
  - Tangential proper part: $TPP(a,b) ::= (a \leq b) \wedge \neg(a << b) \wedge \neg(b \leq a)$
  - Tangential proper part^{-1}: $TPP^{-1}(a,b) ::= (b \leq a) \wedge \neg(b << a) \wedge \neg(a \leq b)$
  - Nontangential proper part: $NTPP(a,b) ::= (a << b) \wedge (a \neq b)$
  - Nontangential proper part^{-1}: $NTPP^{-1}(a,b) ::= (b << a) \wedge (b \neq a)$
  - Equal: $EQ(a,b) ::= (a = b)$

Syntax and relational semantics

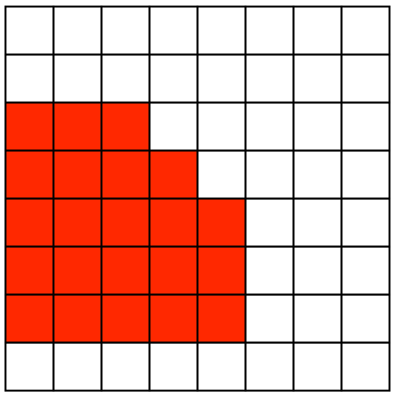
- Relational semantics
  - Frame (adjacency space)
    - Relational system F = (W,R)
      - W: nonempty set (cells)
      - R: binary relation on W (adjacency relation)



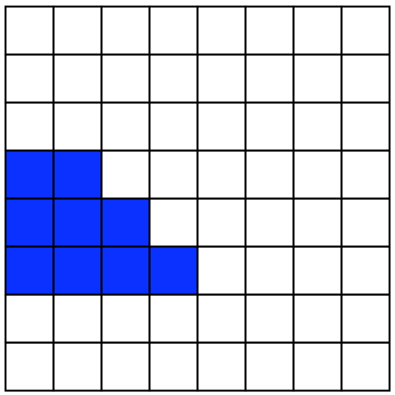
Syntax and relational semantics

- Relational semantics
  - Frame (adjacency space)
    - Relational system F = (W,R)
      - W: nonempty set (cells)
      - R: binary relation on W (adjacency relation)
      - If $a \subseteq W$ then $[R]a ::= \{x \in W: \forall y \in W(xRy \rightarrow y \in a)\}$ is the set of all cells that are necessarily R-adjacent to a-cells

a



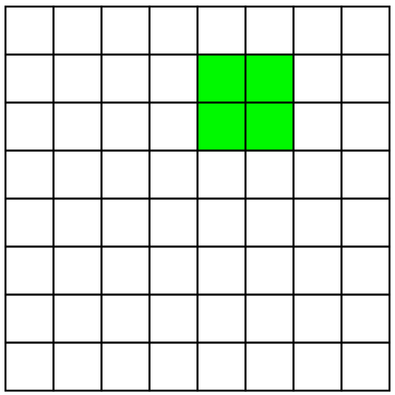
[R]a



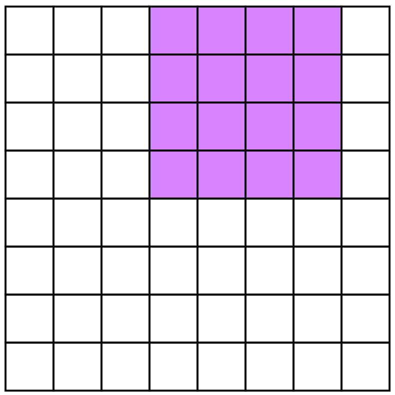
Syntax and relational semantics

- Relational semantics
  - Frame (adjacency space)
    - Relational system F = (W,R)
      - W: nonempty set (cells)
      - R: binary relation on W (adjacency relation)
      - If b⊆W then ⟨R⟩b ::= {x∈W: ∃y∈W(xRy∧y∈b)} is the set of all cells that are possibly R-adjacent to b-cells

b

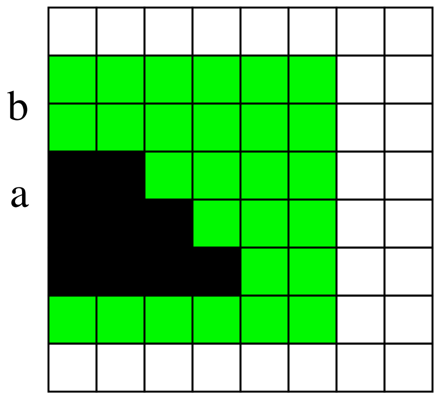


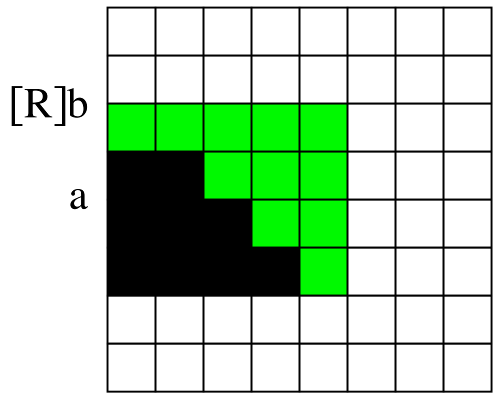
⟨R⟩b



Syntax and relational semantics

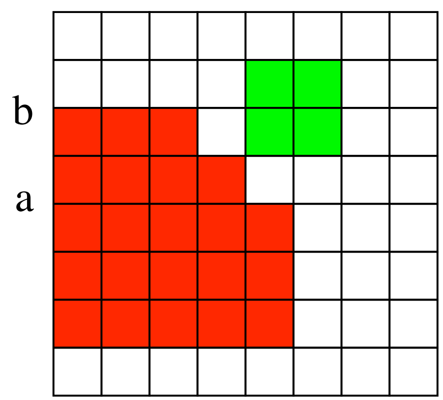
- Relational semantics
  - Regions in an adjacency space F = (W,R)
    - Arbitrary subsets of W
  - Non-tangential inclusion between two subsets a, b
    - $a \ll_R b$ iff for all $x \in W$ and $y \in W$, if $x \in a$ and xRy then $y \in b$
    - $a \ll_R b$ iff $a \subseteq [R]b$

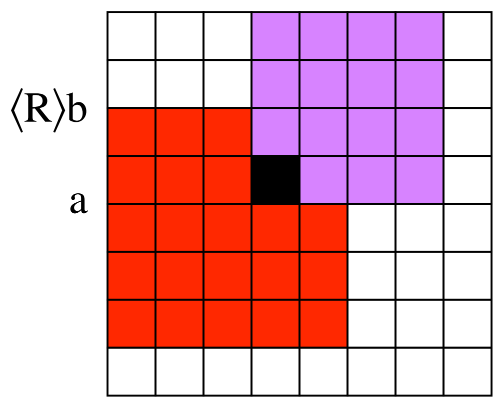




Syntax and relational semantics

- Relational semantics
  - Regions in an adjacency space F = (W,R)
    - Arbitrary subsets of W
  - Contact between two subsets a, b
    - aC_Rb iff for some $x \in W$ and $y \in W$ we have $x \in a$, xRy and $y \in b$
    - aC_Rb iff $a \cap \langle R \rangle b \neq \emptyset$



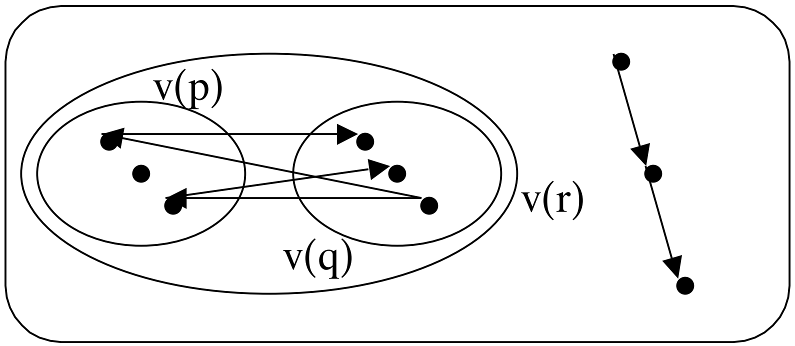


Syntax and relational semantics

- Relational semantics (definition)
  - Valuations in an adjacency space F = (W,R)
    - Functions v assigning to each Boolean variable p a subset v(p) of W
    - $\underline{v}(0) ::= \emptyset$, $\underline{v}(p) ::= v(p)$, $\underline{v}(a^*) ::= W - \underline{v}(a)$, $\underline{v}(a \cup b) ::= \underline{v}(a) \cup \underline{v}(b)$
  - Models over an adjacency space F = (W,R)
    - M = (W,R,v)
  - Truth of modal formulas in a model M = (W,R,v)
    - $M \models (a \leq b)$ iff $\underline{v}(a) \subseteq \underline{v}(b)$, $M \models (aCb)$ iff $\underline{v}(a) C_R \underline{v}(b)$
    - Not $M \models \bot$, $M \models \neg\phi$ iff not $M \models \phi$, $M \models \phi \vee \psi$ iff $M \models \phi$ or $M \models \psi$

Syntax and relational semantics

- Relational semantics (example)
  - Let ϕ be the following modal formula
    - $(p\neq 0)\wedge(q\neq 0)\wedge(r\neq 1)\wedge((p\cup q)=r)\wedge(p\neq r)\wedge(q\neq r)\wedge\neg(pCr^{*})\wedge\neg(qCr^{*})$
  - ϕ is true in the following model



  - ϕ is false in all connected models

Syntax and relational semantics

- Modal logics of classes of frames
  - Logic of a class Σ of frames
    - Set $L(\Sigma)$ of all modal formulas true in Σ
  - **Lemma: If $\Sigma_1 \subseteq \Sigma_2$ then $L(\Sigma_2) \subseteq L(\Sigma_1)$.**
  - Logic of the class Σ_{all} of all frames
    - L_{all}

Syntax and relational semantics

- Modal logics of classes of frames
  - **Lemma: The following modal formulas are true in the class Σ_{all} of all frames:**
    - $(aCb) \rightarrow (a \neq 0)$,
    - $(aCb) \rightarrow (b \neq 0)$,
    - $((a_1 \cup a_2)Cb) \leftrightarrow (a_1Cb) \vee (a_2Cb)$,
    - $(aC(b_1 \cup b_2)) \leftrightarrow (aCb_1) \vee (aCb_2)$.
  - **Lemma: The following modal formulas are true in the class Σ_{wser} of all weakly serial frames:**
    - $(a \neq 0) \leftrightarrow (aC1) \vee (1Ca)$,
    - $(a \leq b) \leftrightarrow \neg((a \cap b^*)C1) \wedge \neg(1C(a \cap b^*))$.

Syntax and relational semantics

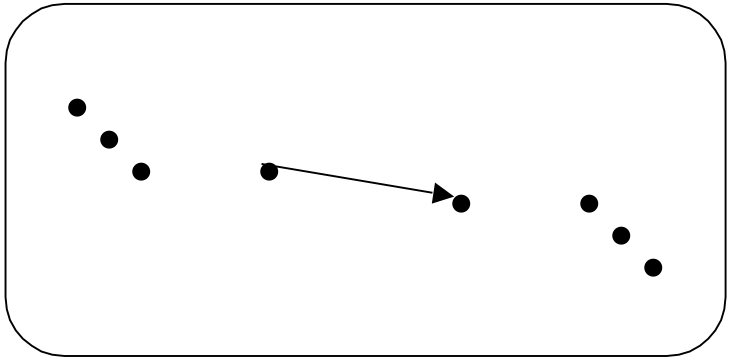
- A translation into modal logic K with universal modality
  - τ: our language $\Rightarrow$ the modal logic K_U
    - $\tau(p) ::= p$
    - $\tau(0) ::= \bot$, $\tau(a^*) ::= \neg\tau(a)$, $\tau(a\cup b) ::= \tau(a)\vee\tau(b)$
    - $\tau(a\leq b) ::= [U](\tau(a)\rightarrow\tau(b))$, $\tau(aCb) ::= \langle U\rangle(\tau(a)\wedge\langle R\rangle\tau(b))$
    - $\tau(\bot) ::= \bot$, $\tau(\neg\phi) ::= \neg\tau(\phi)$, $\tau(\phi\vee\psi) ::= \tau(\phi)\vee\tau(\psi)$
  - **Lemma: $F \models \phi$ (in the sense of our language) iff $F \models \tau(\phi)$ (in the sense of the modal logic K_U).**

Modal definability and undefinability

Modal definability and undefinability

- Modal definability
  - The class Σ of frames is modally definable by the modal formula ϕ iff for every frame $F = (W,R)$, $F \in \Sigma$ iff $F \models \phi$
  - The first-order sentence φ (in R and =) is modally definable by the modal formula ϕ iff for every frame $F = (W,R)$, $F \models \varphi$ iff $F \models \phi$
  - **Theorem: The following decision problem is undecidable:**
    - **Given a first-order sentence φ (in R and =), determine if there exists a modal formula ϕ such that φ is modally definable by ϕ.**

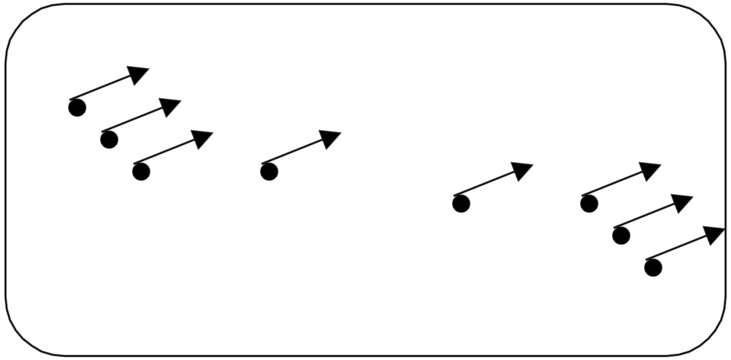
Modal definability and undefinability



- Modal definability
  - **Lemma (first-order examples):**

1. **<u>Non-emptiness of R</u>:** **$(1C1)$.**
2. **Right seriality of R:** **$(p \neq 0) \to (pC1)$.**
3. **Left-seriality of R:** **$(p \neq 0) \to (1Cp)$.**
4. **Weak seriality of R:** **$(p \neq 0) \to (pC1) \vee (1Cp)$.**
5. **Reflexivity of R:** **(Ref) ::= $(p \neq 0) \to (pCp)$.**
6. **Symmetry of R:** **(Sym) ::= $(pCq) \to (qCp)$.**
7. **Universality of R:** **$(p \neq 0) \wedge (q \neq 0) \to (pCq)$.**

Modal definability and undefinability



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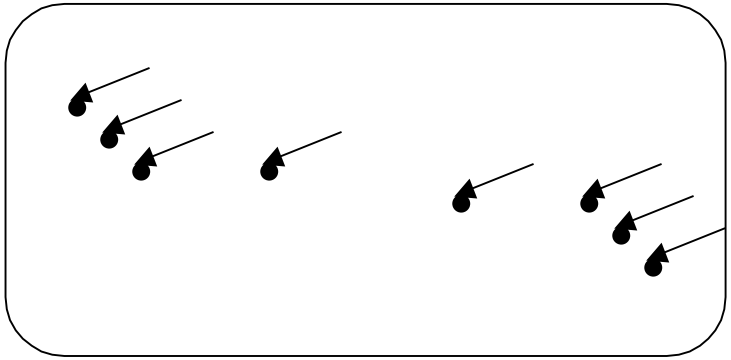
Modal definability and undefinability



- Modal definability
  - **Lemma (first-order examples):**

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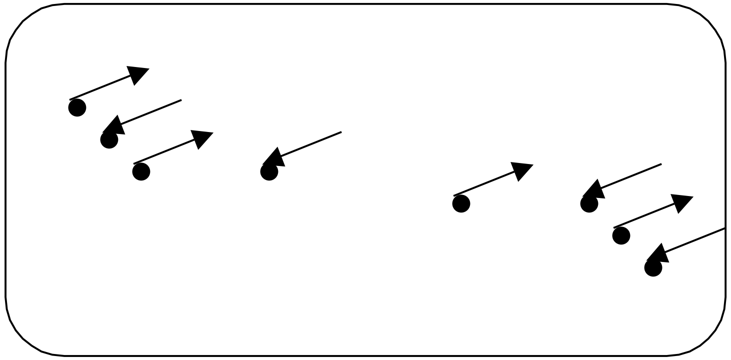
Modal definability and undefinability



- Modal definability
  - **Lemma (first-order examples):**

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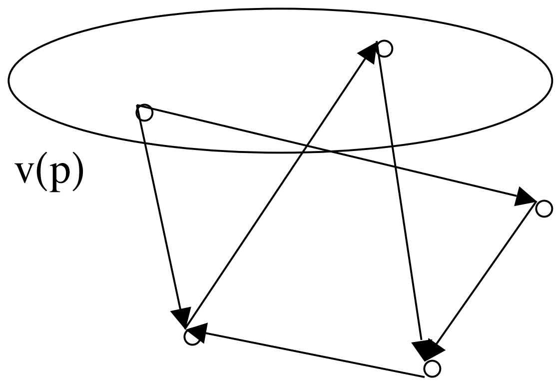
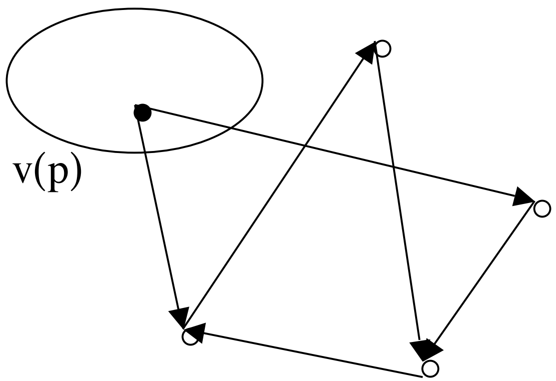
Modal definability and undefinability

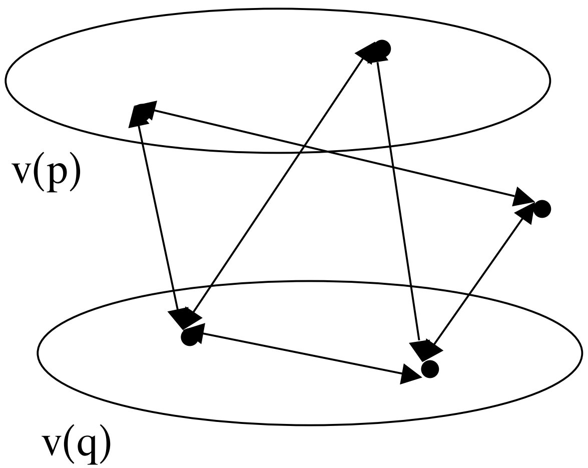
- Modal definability
  - <u>Reflexivity of R</u>: modally defined by (Ref) ::= <u>(p≠0)→(pCp)</u>

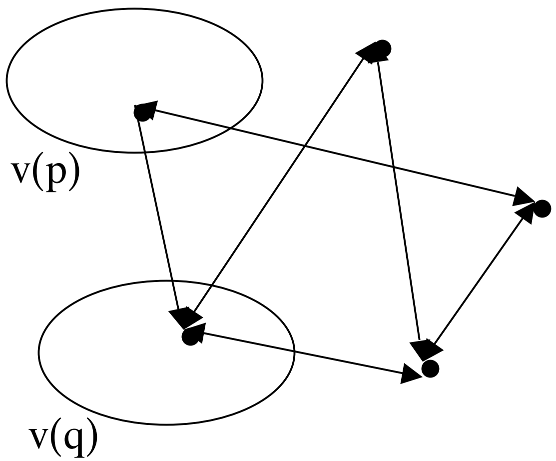




Modal definability and undefinability

- Modal definability
  - Symmetry of R: modally defined by (Sym) ::= (pCq)→(qCp)





Modal definability and undefinability

- Modal definability
  - **Lemma (second-order examples):**

1. **Connectedness of R:**

$$\textbf{(Con) ::=}\ (p \neq 0) \wedge (p^* \neq 0) \rightarrow (pCp^*).$$

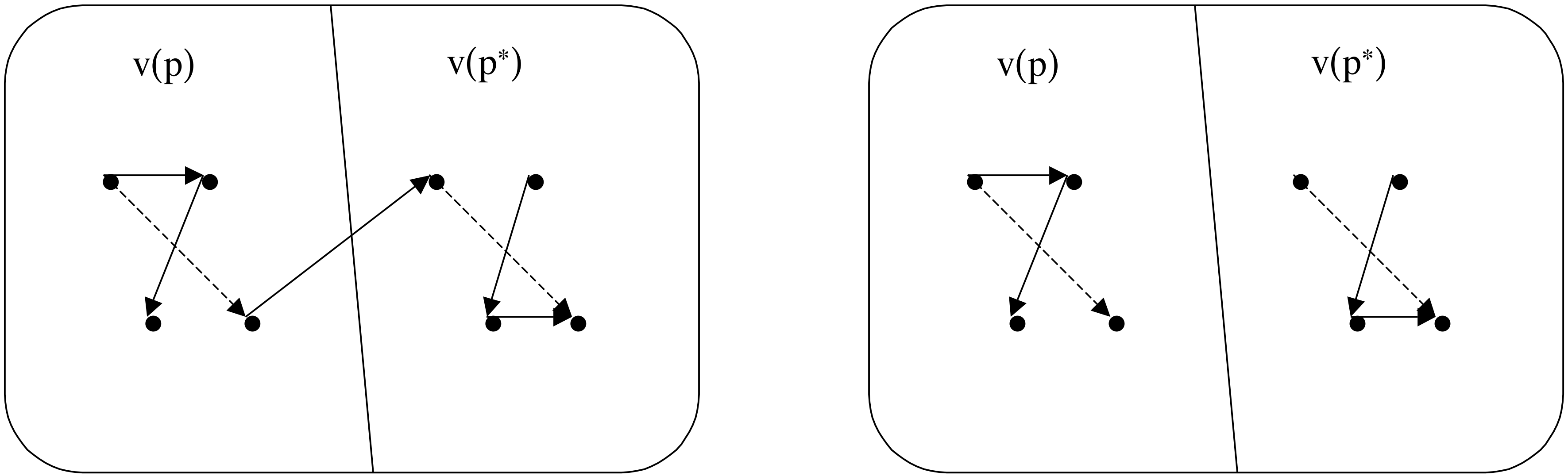
2. **Non n-colourability of R:**

$$\left(\bigcup_{1 \leq i \leq n} p_i = 1\right) \wedge \bigwedge_{1 \leq i < j \leq n} \neg (p_i O p_j) \rightarrow \bigcup_{1 \leq i \leq n} (p_i C p_i).$$

Modal definability and undefinability

- Modal definability
  - Connectedness of R: modally defined by

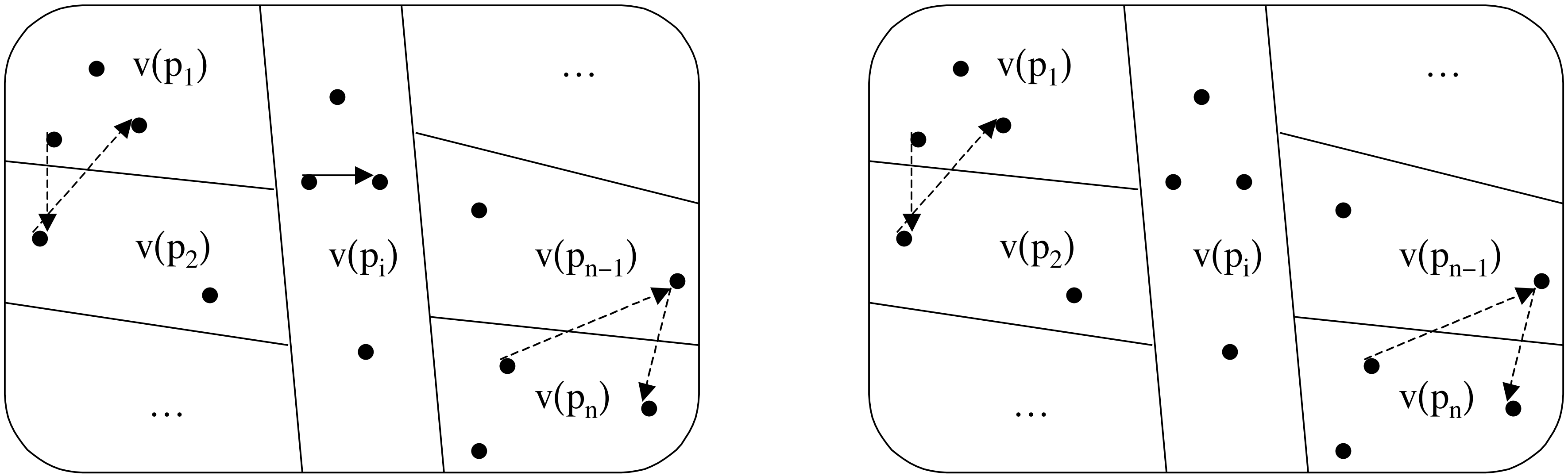
$$(\mathrm{Con}) ::= (p \neq 0) \wedge (p^* \neq 0) \rightarrow (pCp^*)$$



Modal definability and undefinability

- Modal definability
  - Non n-colourability of R: modally defined by

$$(\bigcup_{1\leq i\leq n} p_i = 1) \wedge \bigwedge_{1\leq i<j\leq n} \neg(p_i O p_j) \rightarrow \bigcup_{1\leq i\leq n}(p_i C p_i)$$



Modal definability and undefinability

- Modal undefinability
  - **Lemma (modal undefinability criterion): If $\Sigma_1 \subseteq \Sigma_2$, $\Sigma_1 \neq \Sigma_2$ and $L(\Sigma_1)=L(\Sigma_2)$ then Σ_1 is not modally definable.**
  - Bounded morphism from a model M = (W,R,v) to a model M′ = (W′,R′,v′)
    - Surjective function f from W to W′ such that
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      - If $x'R'y'$ then $f^{-1}(x')C_R f^{-1}(y')$
      - $f(v(p)) \subseteq v'(p)$
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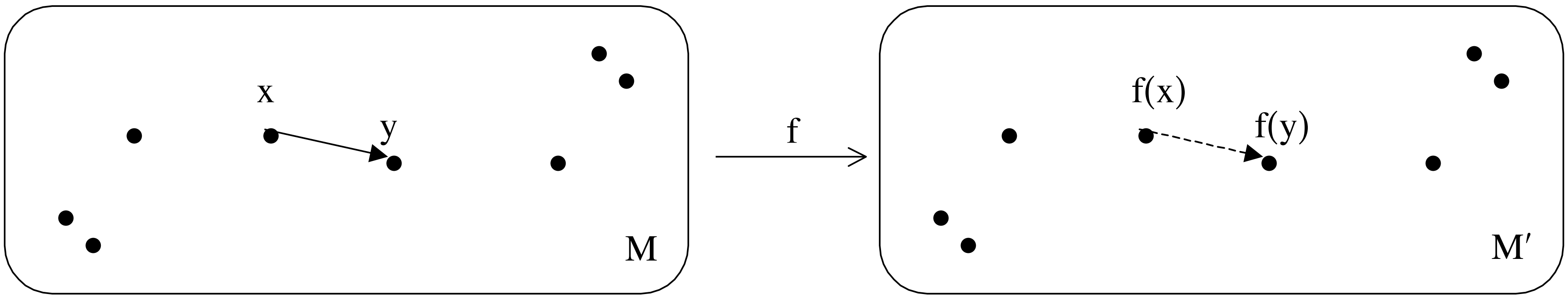

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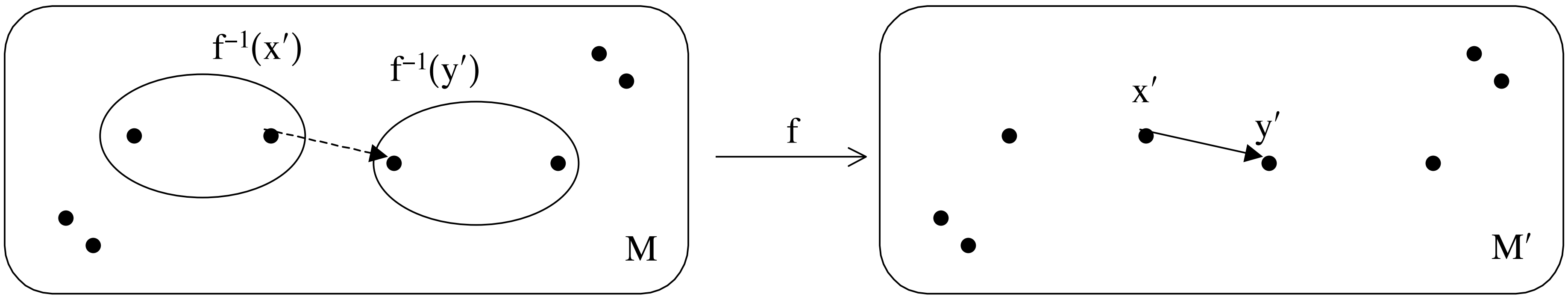

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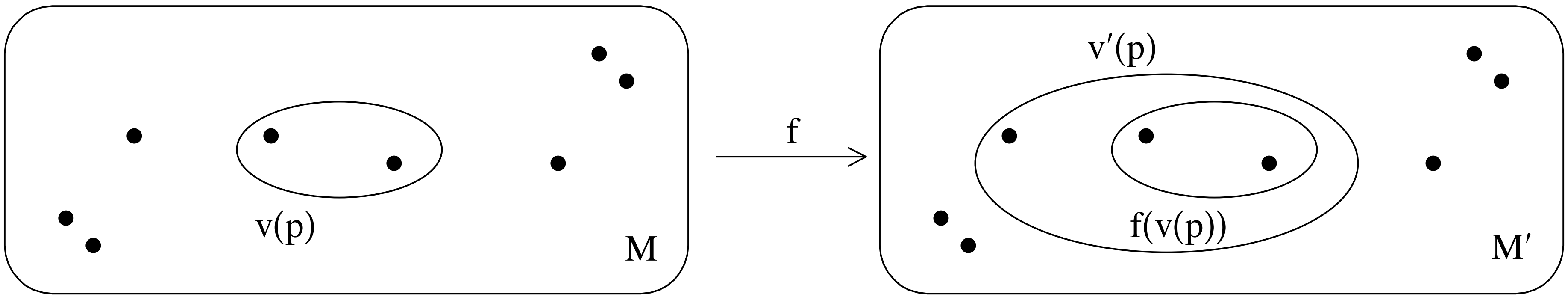

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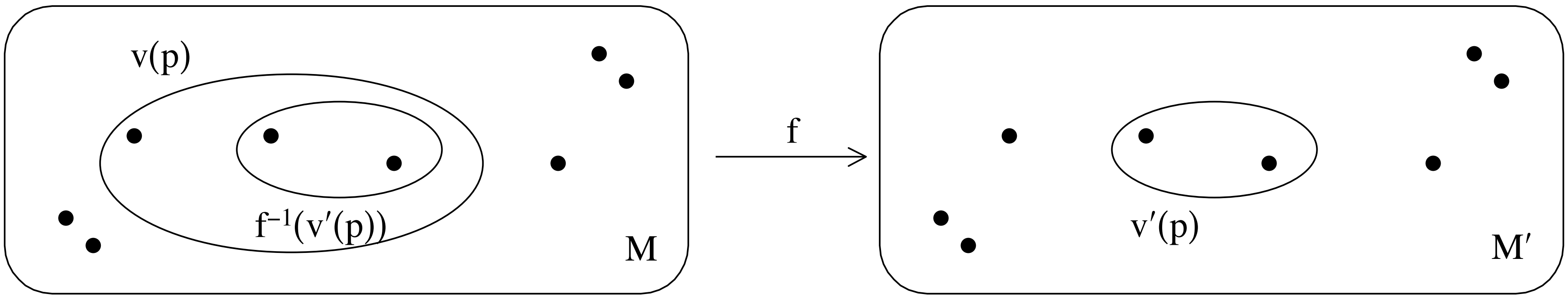

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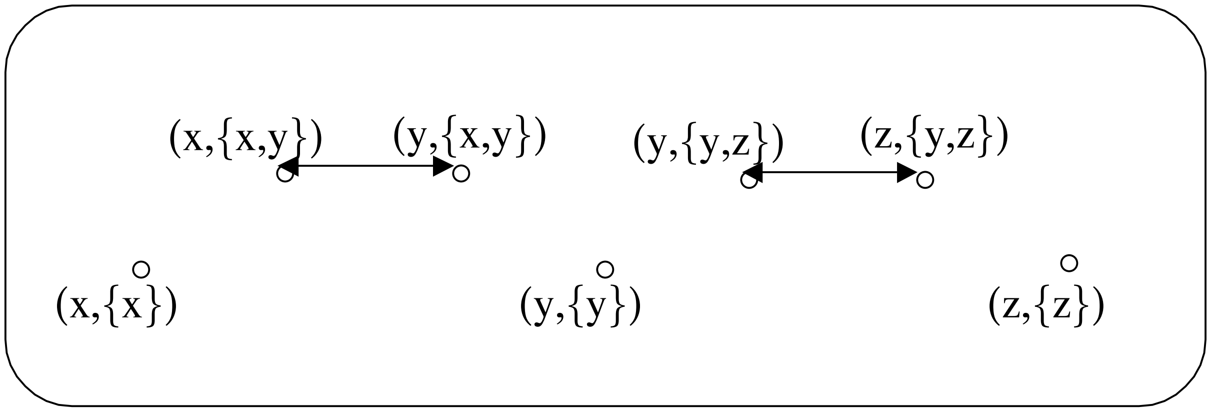
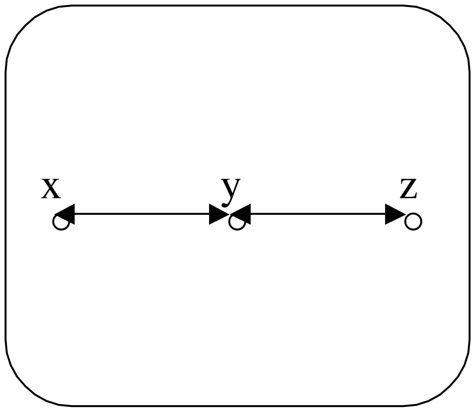
      - $f(v(p)) \subseteq v'(p)$
      - $f^{-1}(v'(p)) \subseteq v(p)$
  - **Lemma (bounded morphism lemma): Let f be a bounded morphism from the model M = (W,R,v) to the model M′ = (W′,R′,v′). $M \models \phi$ iff $M' \models \phi$.**

Modal definability and undefinability

- Modal undefinability
  - **Lemma: Let $\Sigma_{ref,sym}$ be the class of all reflexive and symmetric frames and Σ_e be the class of all equivalence relations.**

1. **$L(\Sigma_{ref,sym}) = L(\Sigma_e)$.**
2. **Σ_e is not modally definable.**



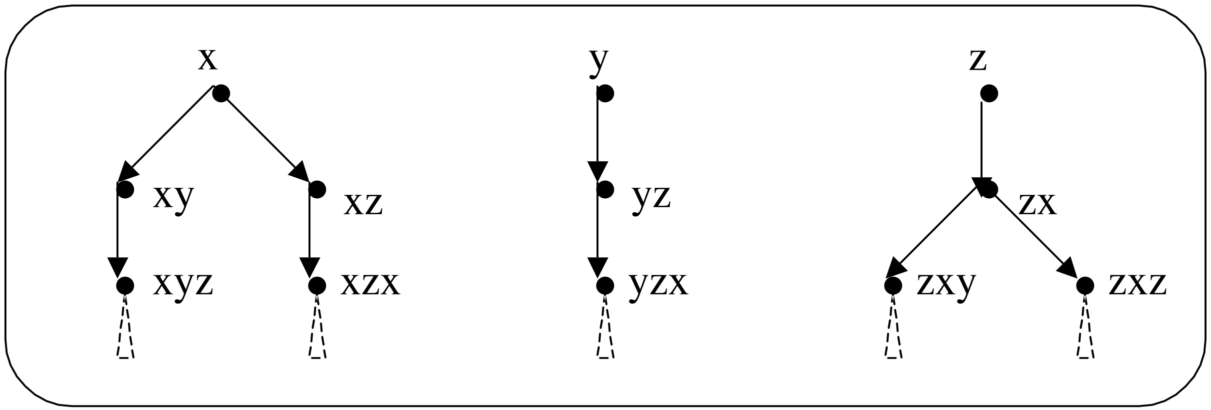


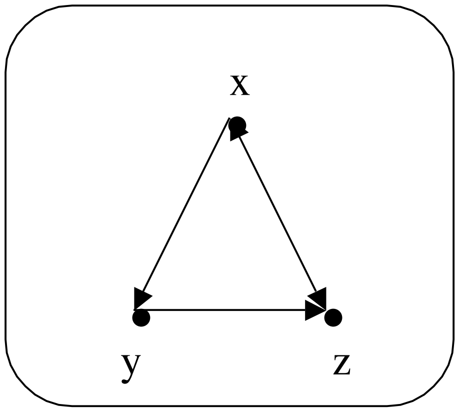
Modal definability and undefinability

- Modal undefinability
  - **Lemma: Let $\Sigma_{2\text{-colour}}$ be the class of all 2-colourable frames.**

1. $L_{all} = L(\Sigma_{2\text{-colour}})$.

2. $\Sigma_{2\text{-colour}}$ is not modally definable.





Axiomatizations and completeness theorems

Axiomatizations and completeness theorems

- Axiomatizations
  - Axiomatic system L_{min} for the logic L_{all}
    - Axioms
      - $(aCb) \to (a \neq 0)$
      - $(aCb) \to (b \neq 0)$
      - $((a_1 \cup a_2)Cb) \leftrightarrow (a_1Cb) \vee (a_2Cb)$
      - $(aC(b_1 \cup b_2)) \leftrightarrow (aCb_1) \vee (aCb_2)$
    - Rules of inference
      - Modus ponens: from $\vdash \phi$ and $\vdash \phi \to \psi$, infer $\vdash \psi$
  - Extensions of L_{min}
    - L_{min}+Ax where Ax is an arbitrary set of axiom schemes
    - L_{min}+R where R is an additional rule of inference
  - **Lemma: There is a continuum of axiomatic extensions of L_{min}.**

Axiomatizations and completeness theorems

- Canonical models
  - Let L be an axiomatic extension of L_{min}
    - L-theory
      - Set of formulas containing all theorems and closed under modus ponens
    - Consistent L-theory
      - L-theory not containing $\perp$
    - Maximal L-theory
      - Consistent L-theory containing ϕ or $\neg\phi$ for each modal formula ϕ
  - **Lemma (Lindenbaum lemma): Any consistent L-theory S can be extended into a maximal L-theory S′.**

Axiomatizations and completeness theorems

- Canonical models
  - Let L be an axiomatic extension of L_{min} and S be a maximal L-theory
    - $a \leq_S b$ iff $(a \leq b) \in S$
      - $a =_S b$ iff $a \leq_S b$ and $b \leq_S a$
    - S-filter
      - Set Γ of boolean terms containing 1 and such that
      1. If $a \in \Gamma$ and $a \leq_S b$ then $b \in \Gamma$
      2. If $a \in \Gamma$ and $b \in \Gamma$ then $a \cap b \in \Gamma$
    - Consistent S-filter
      - S-filter not containing 0
    - Maximal S-filter
      - Consistent S-filter containing a or a^* for each Boolean term a

Axiomatizations and completeness theorems

- Canonical models
  - Let L be an axiomatic extension of L_{min} and S be a maximal L-theory
    - Canonical frame $F_S = (W_S, R_S)$
      - W_S is the set of all maximal S-filters
      - FR_SG iff for all $a \in F$ and $b \in G$ we have $(aCb) \in S$
  - **Lemma (R-extension lemma): Any consistent S-filters F and G such that FR_SG can be extended into maximal S-filters F' and G' such that $F'R_SG'$.**

Axiomatizations and completeness theorems

- Canonical models
  - Let L be an axiomatic extension of L_{min} and S be a maximal L-theory
    - Canonical frame $F_S = (W_S, R_S)$
      - W_S is the set of all maximal S-filters
      - FR_SG iff for all $a \in F$ and $b \in G$ we have $(aCb) \in S$
  - **Lemma (characterization of C and ≤):**

1. **$(a \leq b) \in S$ iff for all $F \in W_S$, if $a \in F$ then $b \in F$.**
2. **$(aCb) \in S$ iff for some $F \in W_S$ and $G \in W_S$ we have $a \in F$, FR_SG and $b \in G$.**

Axiomatizations and completeness theorems

- Canonical models
  - Let L be an axiomatic extension of L_{min} and S be a maximal L-theory
    - Canonical valuation in $F_S = (W_S,R_S)$
      - $v_S(p) ::= \{F \in W_S: p \in F\}$
    - Canonical model over $F_S = (W_S,R_S)$
      - $M_S = (W_S,R_S,v_S)$
  - **Lemma (truth lemma):**

1. $\underline{\mathbf{v}}_{\mathbf{S}}\mathbf{(a) ::= \{F \in W_S: a \in F\}}$.

2. $\mathbf{M_S \models \phi}$ **iff** $\boldsymbol{\phi \in}$ **S.**

  - **Lemma (canonical model lemma): A modal formula $\boldsymbol{\phi}$ is a theorem of L iff $\boldsymbol{\phi}$ is true in all canonical models of L.**

Axiomatizations and completeness theorems

- Completeness theorems
  - **Theorem (completeness theorem for L_{min}):**

1. **Weak completeness. A modal formula ϕ is a theorem of L_{min} iff ϕ is true in all frames.**
2. **Strong completeness. A set S of modal formulas is consistent in L_{min} iff S has a model.**

Axiomatizations and completeness theorems

- Completeness theorems
  - Let L be an axiomatic extension of L_{min}
  - **Proposition (canonical definability lemma):**

1. **∀S, Non-emptiness of R_S: (1C1) is in L.**
2. **∀S, Right seriality of R_S: $(p \neq 0) \rightarrow (pC1)$ is in L.**
3. **∀S, Left-seriality of R_S: $(p \neq 0) \rightarrow (1Cp)$ is in L.**
4. **∀S, Weak seriality of R_S: $(p \neq 0) \rightarrow (pC1) \vee (1Cp)$ is in L.**
5. **∀S, Reflexivity of R_S: (Ref) ::= $(p \neq 0) \rightarrow (pCp)$ is in L.**
6. **∀S, Symmetry of R_S: (Sym) ::= $(pCq) \rightarrow (qCp)$ is in L.**
7. **∀S, Universality of R_S: $(p \neq 0) \wedge (q \neq 0) \rightarrow (pCq)$ is in L.**

Axiomatizations and completeness theorems

- Completeness theorems
  - **Theorem (strong completeness theorem for some extensions of L_{min}): All extensions of L_{min} with axioms from the canonical definability lemma are strongly complete in the corresponding classes of frames.**
  - **Theorem (strong completeness of the logic of equivalence relations): The logic L_{min}+(Ref)+(Sym) is strongly complete in the class Σ_e of all equivalence relations.**

Axiomatizations and completeness theorems

- Weak canonicity
  - An axiomatic extension $L = L_{min}+Ax$ of L_{min} is weakly canonical iff Ax is true in some canonical frame for L
  - **Theorem: Every axiomatic extension of L_{min} is weakly canonical.**

Axiomatizations and completeness theorems

- Strong canonicity
  - An axiomatic extension L = L_{min}+Ax of L_{min} is strongly canonical iff Ax is true in all canonical frames for L
  - **Theorem: All axiomatic extensions of L_{min} with axioms from the canonical definability lemma are strongly canonical.**
  - **Proposition: The logic L_{min}+(Con) is not strongly canonical.**

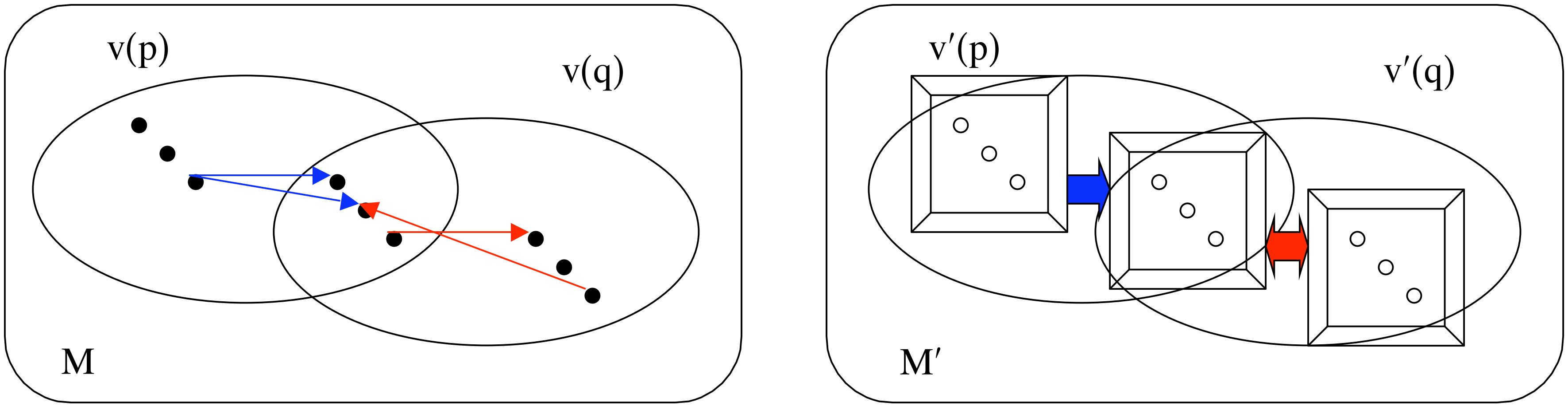
Filtration and small canonical models

Filtration and small canonical models

- Filtration
  - Let M = (W,R,v) be a model and BV be a set of Boolean variables
  - Define the equivalence relation ≡ in W as follows
    - x≡y iff for all p∈BV, x∈v(p) iff y∈v(p)
  - The filtration of M = (W,R,v) through BV is the model M′ = (W′,R′,v′) such that
    - $W' = W_{|\equiv}$
    - $|x| R' |y|$ iff for some z∈W and t∈W we have x≡z, zRt and t≡y
    - For all p∈BV, $v'(p) = v(p)_{|\equiv}$
  - Remark that $Card(W') \leq 2^{Card(BV)}$

Filtration and small canonical models

- Filtration



– **Lemma (filtration lemma):**

1. **For every Boolean term a over BV, $\underline{v}(a)_{|\equiv} = \underline{v'}(a)$.**
2. **For every modal formula ϕ over BV, $M \models \phi$ iff $M' \models \phi$.**

Filtration and small canonical models

- Small canonical models
  - Let $L = L_{min}$+Ax be an axiomatic extension of L_{min}, S be a maximal L-theory, $M_S = (W_S,R_S,v_S)$ be the canonical model corresponding to S and BV be a finite set of Boolean variables
  - Let $M_S' = (W_S',R_S',v_S')$ be the filtration of $M_S = (W_S,R_S,v_S)$ through BV
  - The frame $F_S' = (W_S',R_S')$ is called small canonical frame for L
  - **Lemma (small canonical frame lemma): Ax is true in all small canonical frames for L.**

Filtration and small canonical models

- Weak completeness theorems for the extensions of L_{min}
  - **Theorem: Let $L = L_{min}$+Ax be an axiomatic extension of L_{min}, Σ_{Ax} be the class of all frames determined by Ax and $\Sigma_{Ax,fin}$ be the class of all finite frames determined by Ax. The following conditions are equivalent:**

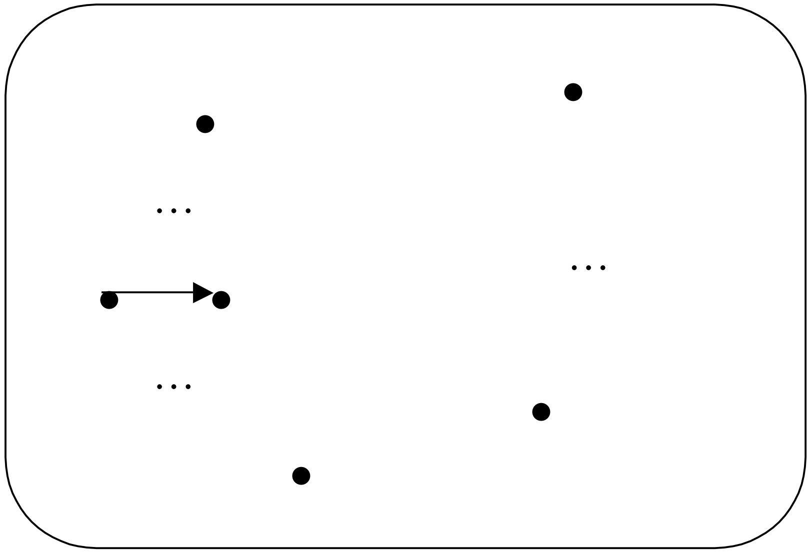
1. **ϕ is a theorem of L.**
2. **ϕ is true in Σ_{Ax}.**
3. **ϕ is true in $\Sigma_{Ax,fin}$.**

Logics related to the colourability of graphs

Logics related to the colourability of graphs

- Logics of non colourability
  - Let L^n be the extension of L_{min} with the axiom scheme
    - $(\bigcup_{1\leq i\leq n} p_i=1) \wedge \bigwedge_{1\leq i<j\leq n} \neg(p_i O p_j) \rightarrow \bigcup_{1\leq i\leq n}(p_i C p_i)$
  - Let L^∞ be $L^1 \cup L^2 \cup \ldots$
  - Note
    - L^1 is L_{min}+<u>(1C1)</u>
    - L^2 is $L_{min}+(pCp) \vee (p^* C p^*)$
    - $L^1 \subset L^2 \ldots \subset L^\infty$

(1C1)



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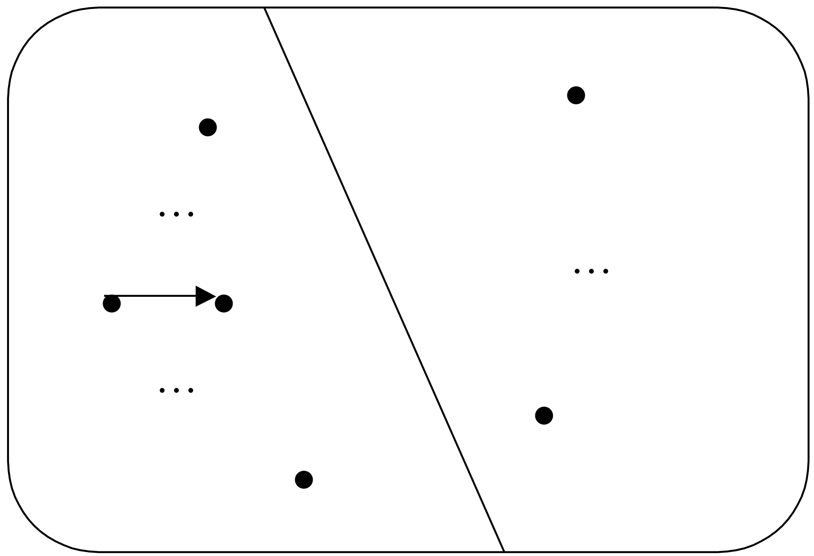

Logics related to the colourability of graphs

- Logics of non colourability
  - **Theorem:**

1. **L^∞ is weakly complete in the class of all finite structures possessing a reflexive point.**
2. **L^∞ is decidable.**
3. **L^∞ is not finitely axiomatizable.**

- **Theorem (strong completeness theorem for L^∞): The logic L^∞ is strongly complete in the class of all frames with a reflexive point.**

Logics related to RCC

Logics related to RCC

- Stell's reformulation of RCC
  - Contact algebra: Boolean algebra $(B,0,*,\cup)$ with a binary relation C of contact such that
    - (RCC1) If aCb then $a \neq 0$ and $b \neq 0$
    - (RCC2) $(a_1 \cup a_2)Cb$ iff a_1Cb or a_2Cb and $aC(b_1 \cup b_2)$ iff aCb_1 or aCb_2
    - (RCC3) If $a \neq 0$ then aCa (the reflexivity axiom)
    - (RCC4) If aCb then bCa (the symmetry axiom)
    - (CON) If $a \neq 0$ and $a^* \neq 0$ then aCa^* (the connectedness axiom)
    - (EXT) If $a \neq 1$ then there exists $b \neq 0$ such that $\neg(aCb)$
  - Additional axiom
    - (NOR) If $\neg(aCb)$ then there exists c such that $\neg(aCc)$ and $\neg(c^*Cb)$

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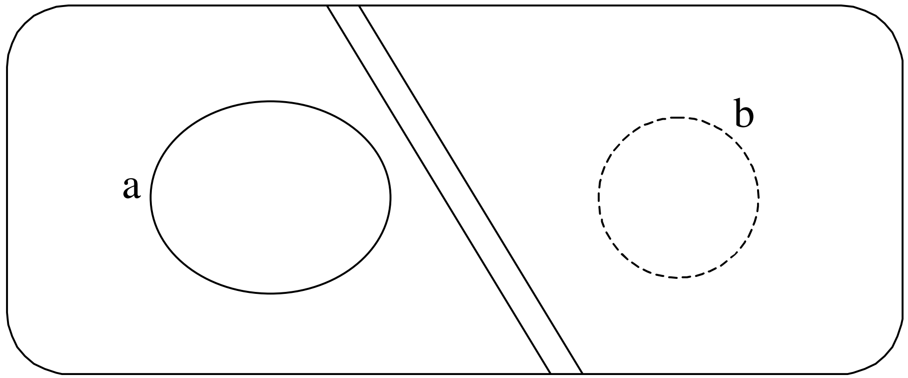

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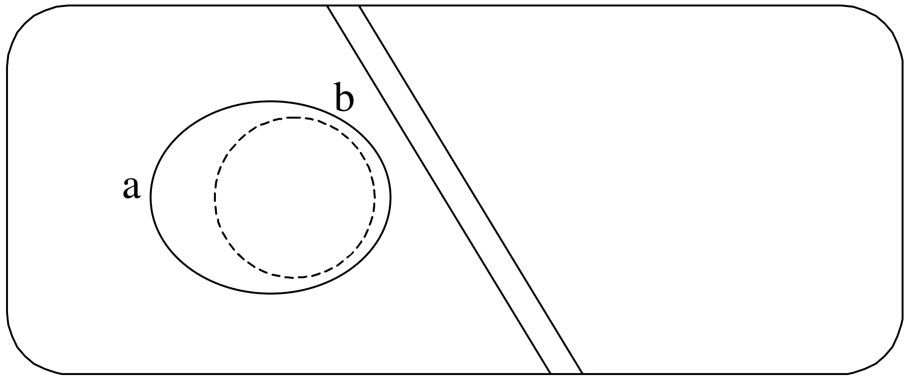

    - (EXT) If $a \neq 0$ then there exists $b \neq 0$ such that $(b \ll a)$
  - Additional axiom
    - (NOR) If $\neg(aCb)$ then there exists c such that $\neg(aCc)$ and $\neg(c^*Cb)$

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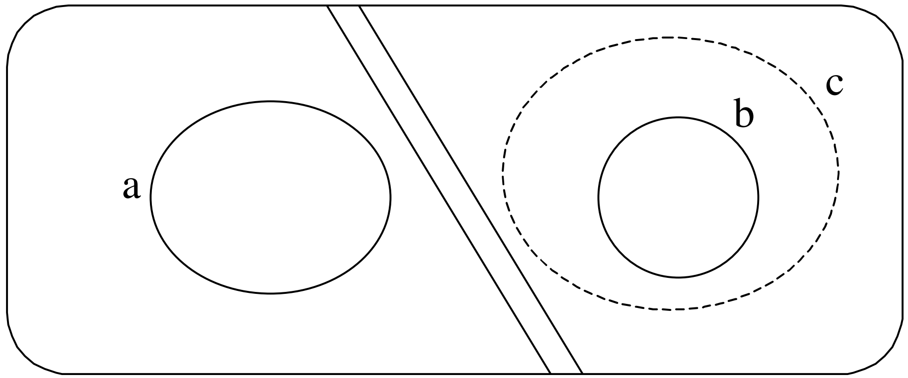

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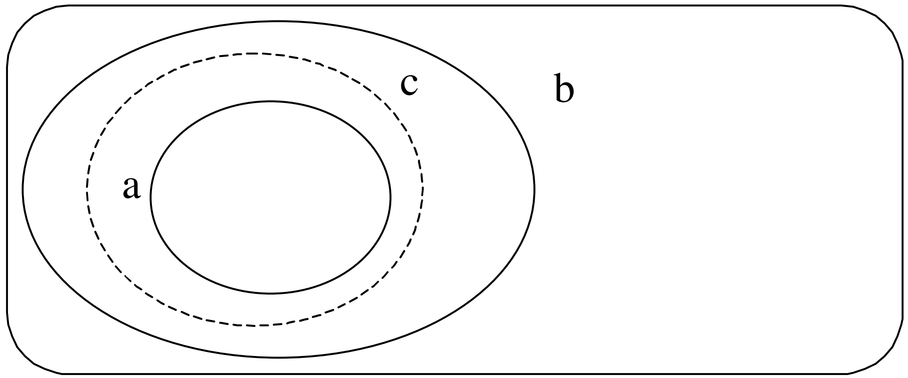
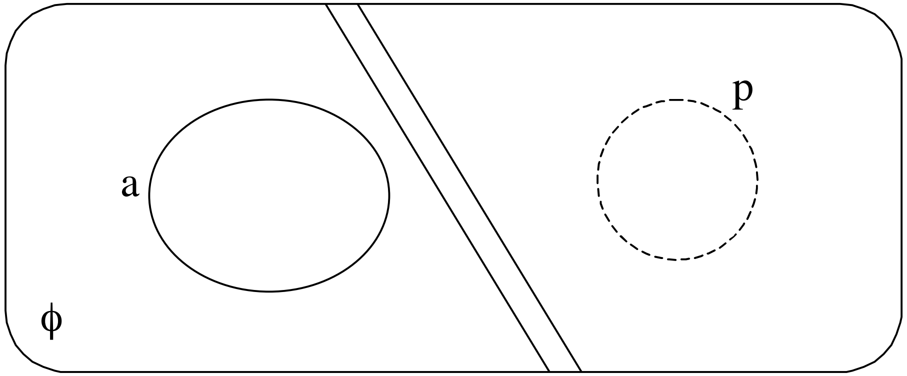


    - (EXT) If a≠0 then there exists b≠0 such that (b<<a)
  - Additional axiom
    - <u>(NOR)</u> <u>If (a<<b) then there exists c such that (a<<c) and (c<<b)</u>

Logics related to RCC

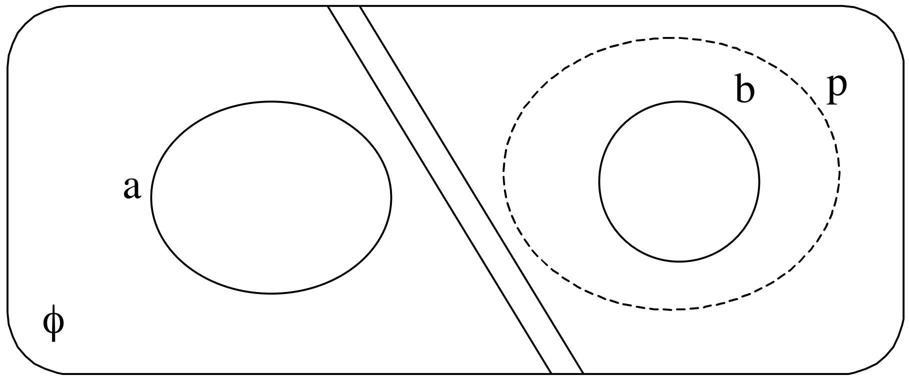
- Let us consider the following systems related to RCC
  - Weak RCC (WRCC): (RCC1)–(RCC4)
  - Connected weak RCC ($WRCC_{CON}$): WRCC+(CON)
  - Extensional weak RCC ($WRCC_{EXT}$): WRCC+(EXT)
  - RCC: WRCC+(CON)+(EXT)
  - Normal extensional weak RCC ($WRCC_{EXT,NOR}$): WRCC+(EXT)+(NOR)
  - Normal RCC (RCC_{NOR}): RCC+(NOR)

Logics related to RCC



- Axioms and rules of inference
  - (Ref): $(p \neq 0) \rightarrow (pCp)$
  - (Sym): $(pCq) \rightarrow (qCp)$
  - (Con): $(p \neq 0) \wedge (p^* \neq 0) \rightarrow (pCp^*)$
  - (Ext): from $\vdash \phi \rightarrow (p=0) \vee (aCp)$ for p a Boolean variable not occurring in $\phi \rightarrow (a=1)$, infer $\vdash \phi \rightarrow (a=1)$
    - (EXT) If $a \neq 1$ then there exists $b \neq 0$ such that $\neg(aCb)$
    - If $\phi \wedge (a \neq 1)$ is consistent then $\phi \wedge (p \neq 0) \wedge \neg(aCp)$ is consistent
  - (Nor): from $\vdash \phi \rightarrow (aCp) \vee (p^*Cb)$ for p a Boolean variable not occurring in $\phi \rightarrow (aCb)$, infer $\vdash \phi \rightarrow (aCb)$
    - (NOR) If $\neg(aCb)$ then there exists c such that $\neg(aCc)$ and $\neg(c^*Cb)$
    - If $\phi \wedge \neg(aCb)$ is consistent then $\phi \wedge \neg(aCp) \wedge \neg(p^*Cb)$ is consistent

Logics related to RCC



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    - If $\phi\wedge(a\neq 1)$ is consistent then $\phi\wedge(p\neq 0)\wedge\neg(aCp)$ is consistent
  - (Nor): from $\vdash \phi\rightarrow(aCp)\vee(p^{*}Cb)$ for p a Boolean variable not occurring in $\phi\rightarrow(aCb)$, infer $\vdash \phi\rightarrow(aCb)$
    - (NOR) If $\neg(aCb)$ then there exists c such that $\neg(aCc)$ and $\neg(c^{*}Cb)$
    - If $\phi\wedge\neg(aCb)$ is consistent then $\phi\wedge\neg(aCp)\wedge\neg(p^{*}Cb)$ is consistent

Logics related to RCC

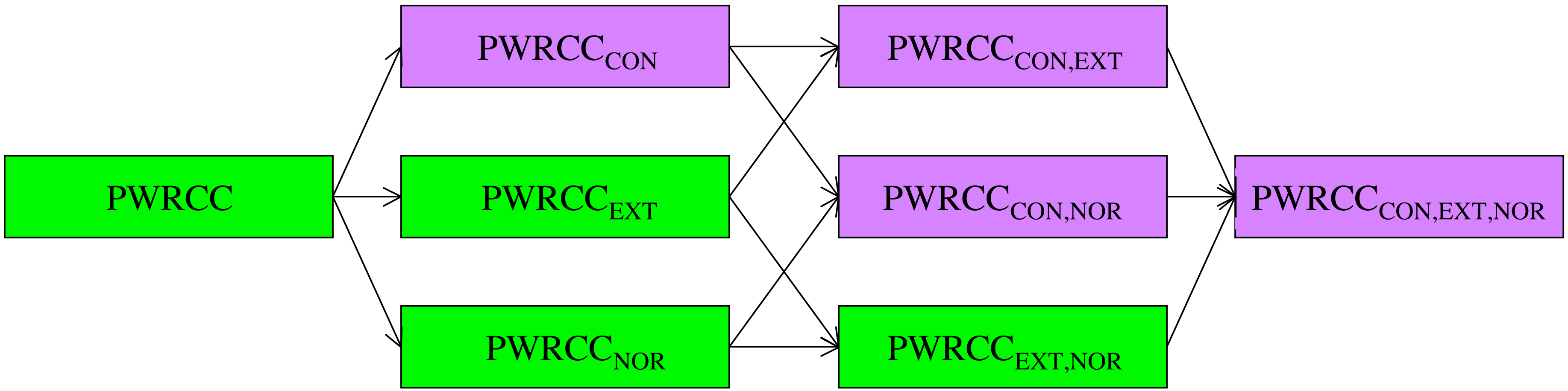
- PWRCC
  - Extension of L_{min} with the axiom schemes (Ref) and (Sym)
- $PWRCC_{EXT}$
  - Extension of PWRCC with the rule of inference (Ext)
- $PWRCC_{NOR}$
  - Extension of PWRCC with the rule of inference (Nor)
- $PWRCC_{EXT,NOR}$
  - Extension of PWRCC with the rules of inference (Ext) and (Nor)

Logics related to RCC

- $PWRCC_{CON}$
  - Extension of L_{min} with the axiom schemes (Ref), (Sym) and (Con)
- $PWRCC_{CON,EXT}$
  - Extension of $PWRCC_{CON}$ with the rule of inference (Ext)
- $PWRCC_{CON,NOR}$
  - Extension of $PWRCC_{CON}$ with the rule of inference (Nor)
- $PWRCC_{CON,EXT,NOR}$
  - Extension of $PWRCC_{CON}$ with the rules of inference (Ext) and (Nor)

Logics related to RCC

- Admissibility of the rules (Ext) and (Nor)
  - **Lemma: (Ext) is an admissibile rule both in PWRCC and also in $PWRCC_{CON}$**
  - **Lemma: (Nor) is an admissibile rule both in PWRCC and also in $PWRCC_{CON}$**



Extensions with rules of inference

Extensions with rules of inference

- The logic $PWRCC_{NOR}$
  - Extension of L_{min} with the axiom schemes (Ref) and (Sym) and the rule of inference (Nor)
    - (Nor): from $\vdash \phi \rightarrow (aCp) \vee (p^{*}Cb)$ for p a Boolean variable not occurring in $\phi \rightarrow (aCb)$, infer $\vdash \phi \rightarrow (aCb)$
- The logic $PWRCC_{NOR\infty}$
  - Extension of L_{min} with the axiom schemes (Ref) and (Sym) and the rule of inference (Nor_{∞})
    - (Nor_{∞}): from $\vdash \phi \rightarrow (aCp) \vee (p^{*}Cb)$ for all Boolean variables p, infer $\vdash \phi \rightarrow (aCb)$

Extensions with rules of inference

- Some remarks on the effects of (Nor) and (Nor_∞)
  - **Lemma (soundness of $PWRCC_{NOR\infty}$ in the class of all equivalence relations): All theorems of $PWRCC_{NOR\infty}$ are true in the class Σ_e of all equivalence relations.**
  - **Lemma: There exists a set S of modal formulas such that**

1. **S has a model in the class $\Sigma_{ref,sym}$,**
2. **S has a model in the class Σ_e,**
3. **S is consistent in $PWRCC_{NOR}$,**
4. **S is not consistent in $PWRCC_{NOR\infty}$.**

Extensions with rules of inference

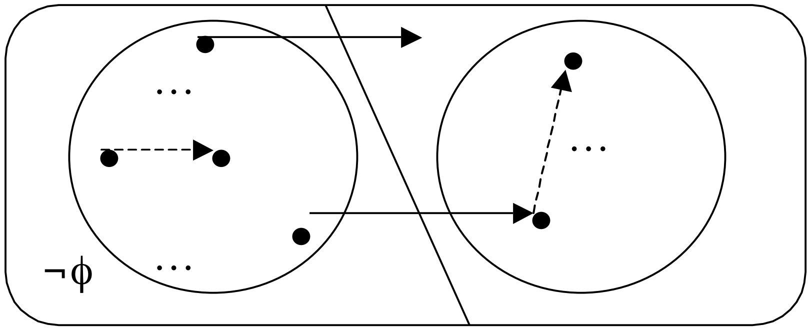
- Some remarks on the effects of (Nor) and (Nor_∞)
  - **Theorem (weak completeness of $PWRCC_{NOR\infty}$ in the class of all equivalence relations): A modal formula ϕ is a theorem of $PWRCC_{NOR\infty}$ iff ϕ is true in the class Σ_e.**
  - **Corollary: The logics $PWRCC_{NOR\infty}$ and L_{min}+(Ref)+(Sym) have the same theorems.**
  - **Proposition: If S is a set of modal formulas consistent in $PWRCC_{NOR\infty}$ then S has a model in Σ_e.**
  - **Proposition: The notion of consistency of $PWRCC_{NOR\infty}$ is not compact.**

Extensions with rules of inference

- Some remarks on the effects of (Nor) and (Nor_{∞})
  - **Lemma: The logics $PWRCC_{NOR\infty}$ and $PWRCC_{NOR}$ have equal sets of theorems.**
  - **Corollary (weak completeness theorem for $PWRCC_{NOR}$): $PWRCC_{NOR}$ is complete in the class Σ_e of all equivalence relations.**
  - **Theorem (strong completeness theorem for $PWRCC_{NOR}$): A set S of modal formulas is consistent in $PWRCC_{NOR}$ iff S has a model in Σ_e.**

Extensions with rules of inference

- The logic of 2-chromatic graphs
  - A frame F = (W,R) is called 2-chromatic if it is not 1-colourable, but is 2-colourable
  - $L_{2\text{-chromatic}}$
    - Extension of L_{min} with the axiom (1C1) and the rule of inference (Col_2)
      - (Col_2): from $\vdash \neg(pCp) \wedge \neg(p^*Cp^*) \rightarrow \phi$ for p a Boolean variable not occurring in ϕ, infer $\vdash \phi$
      - If $\neg\phi$ is consistent then $\neg(pCp) \wedge \neg(p^*Cp^*) \wedge \neg\phi$ is consistent



Extensions with rules of inference

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      - If $\neg\phi$ is consistent then $\neg(pCp)\wedge\neg(p^*Cp^*)\wedge\neg\phi$ is consistent
  - **Lemma: All canonical frames for $L_{2\text{-chromatic}}$ are 2-chromatic.**
  - **Theorem: The logic $L_{2\text{-chromatic}}$ is weakly and strongly complete in the class of all 2-chromatic frames.**
  - **Corollary: The logics $L_{2\text{-chromatic}}$ and L_{min}+(1C1) have the same theorems.**

Some complexity results

Some complexity results

- **Theorem:**

1. **Satisfiability in Σ_{all} is NP-complete.**
2. **Satisfiability in $\Sigma_{ref,sym}$ is NP-complete.**
3. **Satisfiability in the class of all connected frames is PSPACE-complete.**
4. **Satisfiability in the class of all reflexive, symmetric and connected frames is PSPACE-complete.**

Some complexity results

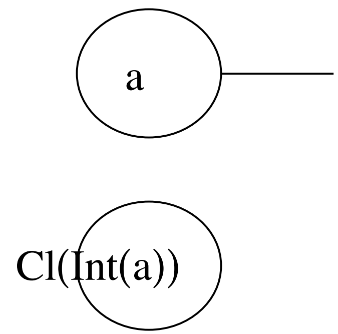
- **Theorem: Let ϕ be a modal formula.**

1. **Satisfiability in the class Σ_ϕ of all frames F = (W,R) such that $F \models \phi$ is in 2EXPTIME.**
2. **If the membership problem in the class Σ_ϕ is in NP then satisfiability in the class Σ_ϕ is in NEXPTIME.**

Topological models

Topological models

- Some topological notions
  - Let X be a topological space
    - $x \in Cl(a)$ iff for all closed sets b of X, if $a \subseteq b$ then $x \in b$
    - $x \in Int(a)$ iff there exists an open set b of X such that $b \subseteq a$ and $x \in b$
  - A subset a of X is <u>regular closed</u> iff $Cl(Int(a)) = a$
  - A subset a of X is regular open iff $Int(Cl(a)) = a$



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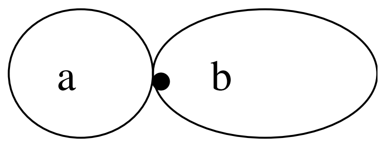
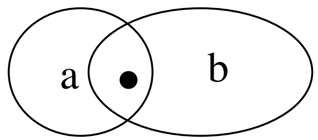
  - A subset a of X is regular closed iff $Cl(Int(a)) = a$
  - A subset a of X is regular open iff $Int(Cl(a)) = a$
  - The algebra $(RC(X), 0, 1, *, \cup, \cap, C)$
    - RC(X) is the set of all regular closed sets of X
    - $0 = \emptyset$, $1 = X$, $a^* = Cl(X - a)$, $a \cup b = a \cup b$, $a \cap b = Cl(Int(a \cap b))$
    - <u>aCb iff $a \cap b \neq \emptyset$</u>





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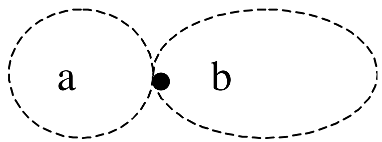
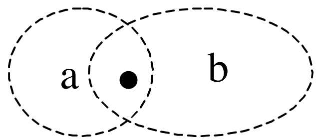
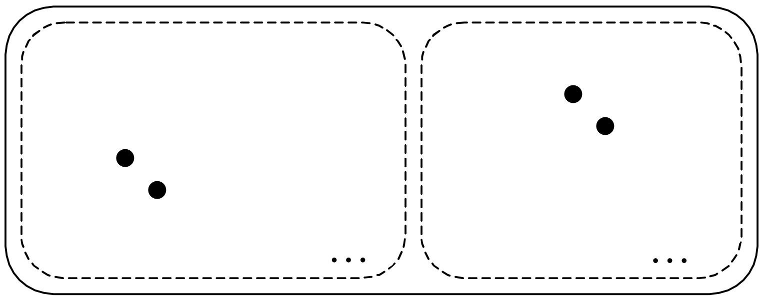
  - The algebra $(RO(X), 0, 1, *, \cup, \cap, C)$
    - $RO(X)$ is the set of all regular open sets of X
    - $0 = \emptyset$, $1 = X$, $a^* = Int(X - a)$, $a \cup b = Int(Cl(a \cup b))$, $a \cap b = a \cap b$
    - <u>aCb iff $Cl(a) \cap Cl(b) \neq \emptyset$</u>





Topological models

- Some topological notions
  - Let X be a topological space
    - X is connected iff X cannot be represented by a sum of two disjoint nonempty open sets of X
    - X is semiregular iff X has a closed base of regular closed sets
    - X is weakly regular iff X is semiregular and for all open sets a of X, there exists an open set b of X such that Cl(a)⊆b
    - X is κ-normal iff every two disjoint regular closed sets of X can be separated by two disjoint open sets of X



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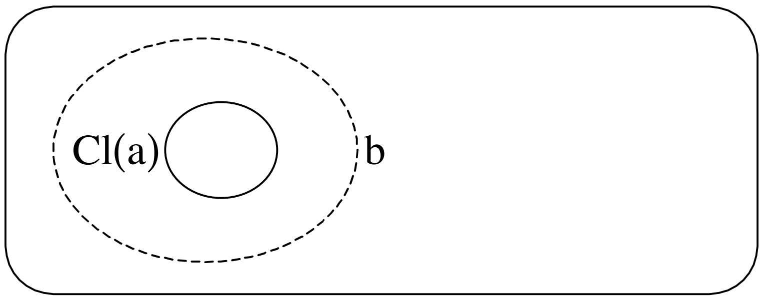

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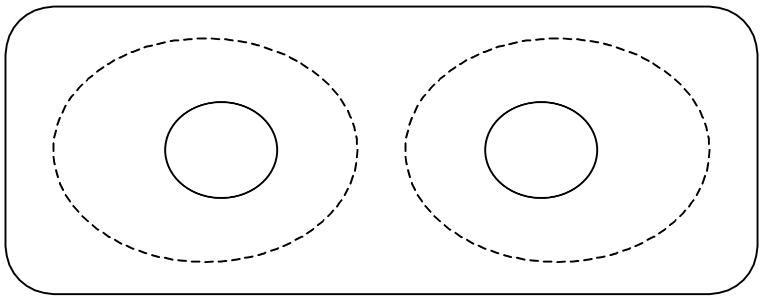


Topological models

- Some topological notions
  - Let X be a topological space
  - **Lemma:**

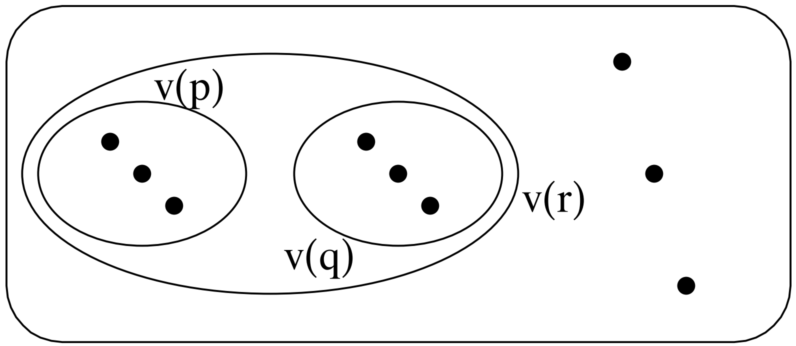
1. **X is connected iff RC(X) satisfies the axiom (CON)**
   - (CON) If $a \neq 0$ and $a^* \neq 0$ then aCa^*
2. **If X is semiregular than X is weakly regular iff RC(X) satisfies the axiom (EXT)**
   - (EXT) If $a \neq 1$ then there exists $b \neq 0$ such that $\neg(aCb)$
3. **X is κ-normal iff RC(X) satisfies the axiom (NOR)**
   - (NOR) If $\neg(aCb)$ then there exists c such that $\neg(aCc)$ and $\neg(c^*Cb)$

Topological models

- Topological semantics (definition)
  - Valuations in a topological space X
    - Functions v assigning to each Boolean variable p a regular closed set v(p) of X
    - $\underline{v}(0) ::= \varnothing$, $\underline{v}(p) ::= v(p)$, $\underline{v}(a^*) ::= Cl(X - \underline{v}(a))$, $\underline{v}(a \cup b) ::= Cl(Int(\underline{v}(a) \cup \underline{v}(b)))$
  - Models over a topological space X
    - M = (X,v)
  - Truth of modal formulas in a model M = (X,v)
    - $M \models (a \leq b)$ iff $\underline{v}(a) \subseteq \underline{v}(b)$, $M \models (aCb)$ iff $\underline{v}(a) \cap \underline{v}(b) \neq \varnothing$
    - Not $M \models \bot$, $M \models \neg\phi$ iff not $M \models \phi$, $M \models \phi \vee \psi$ iff $M \models \phi$ or $M \models \psi$

Topological models

- Topological semantics (example)
  - Let ϕ be the following modal formula
    - $(p\neq 0)\wedge(q\neq 0)\wedge(r\neq 1)\wedge((p\cup q)=r)\wedge(p\neq r)\wedge(q\neq r)\wedge\neg(pCr^{*})\wedge\neg(qCr^{*})$
  - ϕ is true in the following model



  - ϕ is false in all connected models

Topological models

- Modal logics of classes of topological spaces
  - Logic of a class Θ of topological spaces
    - Set $L(\Theta)$ of all modal formulas true in Θ
  - **Lemma: If $\Theta_1 \subseteq \Theta_2$ then $L(\Theta_2) \subseteq L(\Theta_1)$.**
  - Θ_{all}: class of all topological spaces
  - Θ_{con}: class of all connected topological spaces
  - **Lemma (soundness of PWRCC and $PWRCC_{CON}$ with respect to topological semantics):**

1. **All theorems of PWRCC are true in the class Θ_{all}.**
2. **All theorems of $PWRCC_{CON}$ are true in the class Θ_{con}.**

Topological models

- Canonical topological models
  - Let L be an axiomatic extension of PWRCC and S be a maximal L-theory
    - S-clan
      - Set Γ of boolean terms containing 1 and such that
      1. If $a \in \Gamma$ and $a \leq_S b$ then $b \in \Gamma$
      2. If $a \cup b \in S$ then $a \in \Gamma$ or $b \in \Gamma$
      3. If $a \in \Gamma$ and $b \in \Gamma$ then $(aCb) \in S$
    - Maximal S-clan
      - S-clan maximal with respect to set-inclusion

Topological models

- Canonical topological models
  - Let L be an axiomatic extension of PWRCC, S be a maximal L-theory and X_S be the set of all S-clans
  - **Lemma (clan's characterization of C and ≤):**

1. **(a≤b)∈S iff for all $\Gamma \in X_S$, if $a \in \Gamma$ then $b \in \Gamma$.**
2. **(aCb)∈S iff for some $\Gamma \in X_S$ we have $a \in \Gamma$ and $b \in \Gamma$.**

Topological models

- Canonical topological models
  - Let L be an axiomatic extension of PWRCC, S be a maximal L-theory and X_S be the set of all S-clans
    - Define a topology in X_S taking the following subsets (for each Boolean terms a) as a basis for the closed sets
      - $\{\Gamma \in X_S : a \in \Gamma\}$
    - Canonical topological model $M_S = (X_S, v_S)$
      - $v_S(p) ::= \{\Gamma \in X_S : p \in \Gamma\}$
  - **Lemma (truth lemma for the topological semantics):**

1. **$\underline{v}_S(a) ::= \{\Gamma \in X_S : a \in \Gamma\}$.**
2. **$M_S \models \phi$ iff $\phi \in S$.**

Topological models

- Canonical topological models
  - **Lemma (topological canonicity of connectedness): The following conditions are equivalent:**

1. **The axiom (Con) is a theorem of L.**
2. **All canonical topological spaces of L are connected.**

Topological models

- Canonical topological models
  - **Lemma (topological canonicity of extensionality): If L contains the rule (Ext) then all canonical topological spaces of L are extensional.**
  - **Lemma (topological canonicity of normality): If L contains the rule (Nor) then all canonical topological spaces of L are κ-normal.**

Topological models

- Completeness theorems with respect to topological semantics
  - We associate to each logic related to RCC a class of topological spaces
    - PWRCC — All topological spaces
    - $PWRCC_{EXT}$ — All weakly regular topological spaces
    - $PWRCC_{NOR}$ — All κ-normal topological spaces
    - $PWRCC_{EXT,NOR}$ — All κ-normal weakly regular topological spaces
    - $PWRCC_{CON}$ — All connected topological spaces
    - $PWRCC_{CON,EXT}$ — All weakly regular connected topological spaces
    - $PWRCC_{CON,NOR}$ — All κ-normal connected topological spaces
    - $PWRCC_{CON,EXT,NOR}$ — All κ-normal weakly regular connected topological spaces

Topological models

- Completeness theorems with respect to topological semantics
  - **Theorem: The following are equivalent for all modal formulas ϕ:**
    - **ϕ is a theorem of L.**
    - **ϕ is true in all L-spaces.**
    - **ϕ is true in all compact T_0 semiregular L-spaces.**
  - **Theorem: The following are equivalent for all sets S of modal formulas:**
    - **S is consistent in L.**
    - **S has a model in some L-space.**
    - **S has a model in some compact T_0 semiregular L-space.**

Conclusion

Conclusion

- Concluding remarks
  - New kinds of modal logics
    - Discrete models of spatial regions
    - Topological models of spatial regions
  - Two kinds of semantics
    - Relational Kripke-style
    - Topological

Conclusion

- Concluding remarks
  - Relational semantics
    - General definability
    - Sahlqvist's like theory
  - Topological semantics
    - Definability theory
    - Filtration
    - Canonicity

Conclusion

- Future work
  - Variants of part-of and contact in model $M = (W,R,v)$

• Part-of: $M \models (a \leq b)$ iff	$\underline{v}(a) \subseteq \langle R \rangle \underline{v}(b)$	weak part-of
	$\boldsymbol{\underline{v}(a) \subseteq \underline{v}(b)}$	**part-of**
	$\underline{v}(a) \subseteq [R]\underline{v}(b)$	non-tangential inclusion
• Contact: $M \models (aCb)$ iff	$\boldsymbol{\underline{v}(a) \cap \langle R \rangle \underline{v}(b) \neq \emptyset}$	**weak overlap**
	$\underline{v}(a) \cap \underline{v}(b) \neq \emptyset$	overlap
	$\underline{v}(a) \cap [R]\underline{v}(b) \neq \emptyset$	strong overlap

Conclusion

- Future work
  - Weaken the Boolean base
    - Drop the Boolean complement
    - Replace the Boolean axioms with axioms for distributive lattices
  - Introduction of n-ary adjacency relations
    - Relational semantics
      - $C(a_1,\ldots,a_n)$ iff for some $x_1 \in W$, …, $x_n \in W$ we have $x_1 \in v(a_1)$, …, $x_n \in v(a_n)$ and $R(x_1,\ldots,x_n)$
    - Topological semantics
      - $C(a_1,\ldots,a_n)$ iff $v(a_1) \cap \ldots \cap v(a_n) \neq \emptyset$

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