Randomness notions and partial relativization

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Plan of the talk

- Formalizing randomness
- Between 1-randomness and 2-randomness
- Lifting randomness via oracles
- ...and their computability-theoretic counterparts
- Randomness reducibilities $\leq_{LR}$, $\leq_{W2R}$
- Weak 2-randomness in between
- Extras: recent work on weakly 2-randoms

Randomness notions

- Martin-Löf randomness is the most common formalization of randomness
- Certain criticisms have supported stronger notions (2-randomness, weak 2-randomness etc.)

  (left c.e. reals, superlow and other ‘effective’ reals can be Martin-Löf random)
- Martin-Löf randomness interacts best with computability theoretic notions.

Aim of this work

(1) Study randomness between Martin-Löf randomness and 2-randomness.

(2) Provide new interactions of these with computability theory.

Formalizing randomness

- Random sequences should have no special properties
- Random sequences do not belong to certain null sets
- They pass a certain class of statistical sets

Martin-Löf 's abstract approach

- Fix a countable collection of null sets.
- Every sequence that does not belong to any of those sets is called random.
- Random strings have measure 1.

Some randomness notions

- **Martin-Löf randomness**: effectively G_δ sets (Π^0_2 classes) $\cap_i V_i$ such that $\mu V_i < 2^{-i}$.
- Martin-Löf randomness relative to X: replace Π^0_2 with $\Pi^0_2[A]$
- **2-randomness**: $A = \emptyset'$
- **Weak 2-randomness**: Π^0_2 null sets
- **Weak 1-randomness**: Π^0_1 null sets
- **Schnorr randomness**: Π^0_2 null sets $\cap_i V_i$ such that $\mu V_i = 2^{-i}$.

Randomness notions and symbols

Martin-Löf randomness	ML
weak randomness relative to $\emptyset'$	Kurtz[$\emptyset'$]
weak 2-randomness	W2R
Schnorr random relative to $\emptyset'$	SR[$\emptyset'$]
2-randomness	ML[$\emptyset'$]

Strength of notions

$$\boxed{\mathrm{ML}[\emptyset'] \Rightarrow \mathrm{SR}[\emptyset'] \Rightarrow \mathrm{W2R} \Rightarrow \mathrm{Kurtz}[\emptyset'] \cap \mathrm{ML} \Rightarrow \mathrm{ML}}$$

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None of these implications can be reversed.

Lifting randomness via relativization

- Given two classes $\mathcal{M}$ and $\mathcal{N}$, define High($\mathcal{M}, \mathcal{N}$) to be the class containing all oracles A such that $\mathcal{M}^A \subseteq \mathcal{N}$.
- The class of oracles which can lift randomness $\mathcal{M}$ to $\mathcal{N}$.
- For instance, $\mathrm{High}(\mathrm{ML}, \mathrm{SR}[\emptyset'])$ is the set of oracles A such that each set that is Martin-Löf random in A is already $\mathrm{SR}[\emptyset']$.

Computability-theoretic charact. of High($\mathcal{M}, \mathcal{N}$)

Example:

Theorem (Kjos-Hanssen/Miller/Solomon)

Martin-Löf randomness relative to an oracle A is 2-randomness iff A computes an almost everywhere dominating function.

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Martin-Löf randomness relative to an oracle A is 2-randomness iff A computes an almost everywhere dominating function.

$A \in \mathrm{High}(\mathrm{ML}, \mathrm{ML}[\emptyset'])$ *iff A computes an almost everywhere dominating function.*

Partial relativization

- We obtain further characterizations via partial relativizations of standard notions.
- partial relativization was introduced by Simpson in his investigations of mass problems
- ...and has become a useful tool in computability and randomness

Example:

A full relativization of ‘low for random’ gives:

A is low for random relative to B if every B-random is $A \oplus B$-random.

However a more useful and meaningful relation is

every B-random is A-random

We only relativize certain components of a notion.

Computability and partial relativization

- f is **diagonally non-computable** if $f(i) \not\simeq \varphi_i(i)$ for all $i \in \mathbb{N}$.
- C is **d.n.c. by** A if it computes a d.n.c.[A] function
- C is **c.e. traceable by** A if for every $f \leq_T C$ there is A-c.e. family (V_i) with

$$\boxed{f(i) \in V_i \text{ and } |V_i| \text{ computably bounded}}$$

Randomness vs computability theoretic notions

(a) $A \in \mathrm{High}(\mathrm{ML}, \mathrm{Kurtz}[\emptyset'])$ (b) $A \in \mathrm{High}(\mathrm{ML}, \mathrm{W2R})$	$\emptyset'$ is non-d.n.c. by A
(c) $A \in \mathrm{High}(\mathrm{ML}, \mathrm{SR}[\emptyset'])$	$\emptyset'$ is c.e. traceable by A
(d) $A \in \mathrm{High}(\mathrm{W2R}, \mathrm{ML}[\emptyset'])$ (e) $A \in \mathrm{High}(\mathrm{ML}, \mathrm{ML}[\emptyset'])$	A is u.a.e. dominating
(f) $A \in \mathrm{High}(\mathrm{Kurtz}, \mathrm{ML})$	impossible

Randomness reducibilities

- A natural extension of Turing reducibility is $\leq_{LR}$
- $A \leq_{LR} B$ if every Martin-Löf random relative to B is also random relative to A
- $\ldots$ if $B \in \mathrm{High}(\mathrm{ML}, \mathrm{ML}^A)$
- Intuitively, B can derandomize all sequences that A can.
- $A \equiv_{LR} B$ if the class of Martin-Löf randoms relative to A coincides with the class of Martin-Löf randoms relative to B

Reducibility associated with weak 2-randomness

- The reducibility associated with weak 2-randomness is $\leq_{W2R}$.
- $A \leq_{W2R} B$ if every weakly 2-random relative to B is also weakly 2-random relative to A.

Open problem

Proposition (Kjos-Hanssen, Kučera, Nies)

$A \leq_{LR} B$ iff every $\Sigma^0_1(A)$ class of measure < 1 is contained in a $\Sigma^0_1(B)$ class of measure < 1.

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Is there an analogous characterization for $A \leq_{W2R} B$?

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Is there an analogous characterization for $A \leq_{W2R} B$?

Is $A \leq_{W2R} B$ equivalent to every $\Pi^0_2(A)$ null class is contained in some $\Pi^0_2(B)$ null class?

$\leq_{LR}$ versus $\leq_{W2R}$

Theorem

- *$\leq_{W2R}$ implies $\leq_{LR}$*
- *They coincide on the initial segment of low for Ω sets*
- *They coincide on the Δ^0_2 sets.*
- *They do not coincide on the Δ^0_3 sets.*
- *$\equiv_{W2R}$ and $\equiv_{LR}$ coincide.*

Weak 2-randomness between ML and ML[$\emptyset'$]

$$\boxed{\text{1-random}} \Rightarrow \boxed{\text{weak 2-random}} \Rightarrow \boxed{\text{2-random}}$$

Informal question:

> *Is weak 2-randomness closer to to 1-randomness or 2-randomness?*

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The definition of W2R *is a slight modification of the definition of* ML.

Closer to 1-randomness: results

- $A \in$ W2R iff $A \in$ ML and forms a minimal pair with $\emptyset'$ (Hirschfeldt/Miller)

- Lifting ML to W2R is much easier than lifting W2R to ML[$\emptyset'$]

  ...making $\emptyset'$ non-dnc by A is easier than making A a.e. dominating

  ...making a Δ^0_2 set non-low is easier than making it a.e. dominating.

- There is a weakly 2-random which is K-trivial relative to $\emptyset'$.

Two open problems from Nies' book

Problem 8.2.14 Is every weakly 2-random array computable?

Problem 3.6.9 To what extend does van Lambalgen's theorem hold for weak 2-randomness?

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Recent work of Barmpalias/Downey/Ng answers these questions

Theorem (Barmpalias/Downey/Ng)

For every function f there is a weakly 2-random X and a function $g \leq_T X$ which is not dominated by f.

Corollary (Barmpalias/Downey/Ng)

There is an array non-computable weakly 2-random set.

Jumps of randoms

- Recent work includes jump inversion theorems for weakly 2-randoms and 2-randoms
- ...aiming at a full characterization of their jumps
- this work has the following corollary:

Theorem (Barmpalias/Downey/Ng)

If A is weakly 2-random relative to B and B is weakly 2-random then $A \oplus B$ is weakly 2-random. But not vise-versa.

References

Barmpalias/Miller/Nies, Randomness notions and partial relativization, submitted.

Nies, Computability and Randomness, Oxford Press 2009

Downey and Hirschfeldt, Algorithmic randomness and complexity, Springer-Verlag, to appear.

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