Ultrapowers of operator algebras

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Applications in logic, nonstandard analysis, number theory, combinatorics, geometry of Banach spaces,...

Theorem

Assume the Continuum Hypothesis, CH. If A is a structure of cardinality $\leq 2^{\aleph_0}$ then all ultrapowers of A are isomorphic. □

Theorem (Dow, Shelah, 1984)

IF CH fails and A is an infinite linear ordering then A has nonisomorphic ultrapowers.

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(ii) $M_n(\mathbb{C})$.

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A von Neumann algebra is subalgebra of $\mathcal{B}(H)$ closed in the weak operator topology.

It is *tracial* if there is $\tau\colon M \to \mathbb{C}$ such that $\tau(ab) = \tau(ba)$, it is continuous, and *faithful*: $\tau(a^*a) = 0$ if and only if $a = 0$.

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Examples: (i) $L^\infty([0,1], \lambda)$, $\tau(f) = \int f\, d\lambda$.
(ii) $M_n(\mathbb{C})$, $\tau(a) = \frac{1}{n}\text{trace}(a)$.

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Relative commutant

$M \hookrightarrow M^{\mathcal{U}}$ via the diagonal embedding $a \mapsto (a, a, a, \dots)/\mathcal{U}$.

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Question (Kirchberg, 2002)

Is $F_{\mathcal{U}}(\mathcal{B}(H)) = \mathbb{C} \cdot 1$*?*

The following (until further notice) is joint with N. Christopher Phillips and Juris Steprāns

Theorem (FPS)

Assume $\mathcal{V}$ is a selective ultrafilter. Then for $a \in \mathcal{B}(H)^{\mathcal{V}}$ the following are equivalent.

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1. $a \in \mathcal{B}(H)'$.
2. *a has a representing sequence (b_n) that is a norm-central sequence:*

$$\lim_n \|[c, b_n]\| = 0 \text{ for all } c \in M.$$

A solution to Kirchberg's problem

Proposition (FPS, Sherman, folklore(?))

$\mathcal{B}(H)$ *has no nontrivial norm-central sequences.*

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CH implies that $F_{\mathcal{V}}(\mathcal{B}(H))$ is trivial for some $\mathcal{V}$.

A solution to $\frac{1}{4}$ of Kirchberg's problem

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Fix a decomposition

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$$\mathcal{D}[\vec{E}] = \{a \in \mathcal{B}(H) : (\forall i) a[E_i] \subseteq E_i\}.$$

Lemma

1. *If $\vec{E}$ is coarser than $\vec{F}$, then $\mathcal{D}[\vec{E}] \supseteq \mathcal{D}[\vec{F}]$.*
2. $\bigcup_{\vec{E}} \mathcal{D}[\vec{E}] \neq \mathcal{B}(H)$.

A useful lemma

Lemma (Farah, 2007)

$(\forall a \in \mathcal{B}(H))(\forall \varepsilon > 0)(\exists \vec{E}, \vec{F})$

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$(\forall a \in \mathcal{B}(H))(\forall \varepsilon > 0)(\exists \vec{E}, \vec{F})$

$$a = a_{\vec{E}} + a_{\vec{F}} + c$$

where $a_{\vec{E}} \in \mathcal{D}[\vec{E}]$, $a_{\vec{F}} \in \mathcal{D}[\vec{F}]$, *$c$ is compact and* $\|c\| < \varepsilon$.

What's in the commutant?

Lemma

For any $\mathcal{U}$ we have

$$\mathcal{B}(H)' \cap \mathcal{B}(H)^{\mathcal{U}} = \bigcap_{\vec{E}} \mathcal{D}[\vec{E}]' \cap \mathcal{B}(H)^{\mathcal{U}}.$$

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Proof.

$\subseteq$ is trivial.

$\supseteq$: If $a \in$LHS, write $a = a_{\vec{E}} + a_{\vec{F}} + c$. For $b \in \mathcal{B}(H)$ we have

$$\|[b, a]\| = \|[b, c]\| \leq \varepsilon \|b\|$$

for an arbitrarily small $\varepsilon > 0$. □

Flat ultrafilters

Definition (FPS)

An ultrafilter $\mathcal{U}$ is *flat* if there are $h_n \colon \mathbb{N} \searrow [0,1]$ such that

1. $h_n(0) = 1$,
2. $\lim_j h_n(j) = 0$,
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For each n,

$$a_n = \sum_j h_n(j) \operatorname{proj}_{\mathbb{C}\xi_j}$$

is in $\mathcal{B}(H)$.

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If an ultrafilter $\mathcal{U}$ is flat then $F_{\mathcal{U}}(\mathcal{B}(H)) \neq \mathbb{C}$.

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we have $\mathbf{a} \in \mathcal{D}[\vec{E}]'$ for all $\vec{E}$. □

The existence of flat ultrafilters

Theorem (FPS)

There exists a flat ultrafilter on some countable set $\mathbb{F}$.

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$$\mathbb{F} = \{h \colon \mathbb{N} \searrow \mathbb{Q} \cap [0,1] : h(0) = 1, \text{ and } (\forall^\infty m) h(m) = 0\}.$$

For $f : \mathbb{N} \nearrow \mathbb{N}$ and $\varepsilon > 0$ let

$$\mathbf{X}_{f,\varepsilon} = \{h \in \mathbb{F} : \|h - h \circ f\|_\infty \leq \varepsilon.$$

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If $n > 1/\varepsilon$ then

$$h = \chi[0, f(0)) + \frac{n-1}{n}\chi_{[f(0), f^2(0)))} + \cdots + \frac{1}{n}\chi_{[f^{n-1}(0), f^n(0))}$$

belongs to $\mathbf{X}_{f,\varepsilon}$.

Hence each $\mathbf{X}_{f,\varepsilon}$ is infinite, and

$$\mathbf{X}_{f,\varepsilon} \cap \mathbf{X}_{g,\delta} \supseteq \mathbf{X}_{\max(f,g),\min(\varepsilon,\delta)}.$$

An ultrafilter on $\mathbb{F}$ containing all $\mathbf{X}_{f,\varepsilon}$ is flat. $\square$

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2. *Assuming CH or Martin's Axiom, there is an ultrafilter* $\mathcal{U}$ *such that* $F_{\mathcal{U}}(\mathcal{B}(H)) = \mathbb{C}$.

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Assuming CH or Martin's Axiom $F_{\mathcal{U}}(\mathcal{B}(H))$ *depends on the choice of the ultrafilter.*

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Assuming CH or Martin's Axiom $F_{\mathcal{U}}(\mathcal{B}(H))$ *depends on the choice of the ultrafilter.*

What about the remaining 1/4 of the problem?

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If ZFC is consistent, then so is 'ZFC+there are no selective ultrafilters.'

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Theorem (Kunen, 1976)

If ZFC is consistent, then so is 'ZFC+there are no selective ultrafilters.'

We can do with a P-point instead of a selective ultrafilter, but Shelah proved that consistently there are no P-points.

ε-flatness

Definition

An ultrafilter $\mathcal{U}$ is ε-*flat* for some $\varepsilon > 0$ if there are $h_n \colon \mathbb{N} \searrow [0,1]$ such that

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3. $(\forall f : \mathbb{N} \nearrow \mathbb{N}) \lim_{n \to \mathcal{U}} \|h_n - h_n \circ f\|_\infty \leq \varepsilon$.

Fact

$\mathcal{U}$ is flat iff a single sequence (h_n) witnesses ε-flatness of $\mathcal{U}$ for all $\varepsilon > 0$.

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Proposition (FS)

Assume there are no P-points. Then every ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is ε-flat for every $\varepsilon > 0$.

Questions

Question

Is there a nonprincipal ultrafilter $\mathcal{V}$ *on* $\mathbb{N}$ *such that* $F_{\mathcal{V}}(\mathcal{B}(H)) = \mathbb{C}$*?*

In many models of ZFC the answer is positive.

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Question

Is $F_{\mathcal{V}}(\mathcal{B}(H)) \neq \mathbb{C}$ equivalent to '$\mathcal{V}$ is flat'?

Question (Kirchberg, 2004)

Does $F_{\mathcal{U}}(M)$ depend on M for a separable C-algebra M?*

Separable C*-algebras

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Proposition (Ge–Hadwin, 2001)

CH implies that for all $\mathcal{U}$ and $\mathcal{V}$

$$F_{\mathcal{U}}(M) \cong F_{\mathcal{V}}(M)$$

for every separable C-algebra M.*

Theorem (F., 2008)

Con(ZFC) *implies* Con(ZFC $+$ *there are* $\mathcal{U}$ *and* $\mathcal{V}$ *such that*

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for some separable C-algebra* M*).*

A consequence of the Dow–Shelah result

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$\neg$CH implies there are $\mathcal{U}$ and $\mathcal{V}$ such that

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for every separable C-algebra M that has an infinite chain of projections.*

The remaining results are joint with Bradd Hart and David Sherman

Theorem (FHS, 2009)

Assume CH fails. For a countable metric structure A the following are equivalent.

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Theorem (FHS, 2009)

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1. *All ultrapowers of A are isomorphic.*
2. *The theory of A is stable (in a variant of the Ben Yaacov–Berenstein–Henson–Usvyuatsov's 'logic of metric structures').*

Theorem (FHS, 2009)

For every infinite-dimensional separable C-algebra M TFAE:*

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2. *All $F_{\mathcal{U}}(A)$ are isomorphic,*
3. *CH.*

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5. *All relative commutants of the unitary group of A in its ultrapowers are isomorphic.*

The theory of (discrete) abelian groups is stable (Szmielew, 1955).

II_1 factors

Example

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Definition

A von Neumann algebra $(M, +, \cdot, *, \|\cdot\|)$ is *tracial* if it has a faithful normalized trace τ.

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A von Neumann algebra $(M, +, \cdot, *, \|\cdot\|)$ is *tracial* if it has a faithful normalized trace τ.

Example

$L^\infty([0,1], \lambda)$, with $\tau(f) = \int f \, d\lambda$.

II_1 factors

Example

Fix n. Let τ denote the normalized trace on $M_n(\mathbb{C})$.
The ℓ^2-norm (Hilbert–Schmidt norm) on $M_n(\mathbb{C})$:

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Definition

A tracial von Neumann algebra is a *type II_1 factor* if its center is trivial and it is infinite-dimensional.

Question (Dusa McDuff, 1970)

If M is a II_1 factor, are all

$$M' \cap M^{\mathcal{U}}$$

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Theorem (Ge–Hadwin, 2001)

For any separable II_1 factor the Continuum Hypothesis implies all of its ultrapowers are isomorphic, and all of the associated relative commutants are isomorphic.

Theorem (FHS, 2009)

TFAE for every separable II_1 factor M:

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Non-stability is witnessed by $\varphi(x_1, x_2, y_1, y_2)$:

$$\|x_1 y_2 - y_2 x_1\|_2.$$

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An ultraproduct of $M_n(\mathbb{C})$, for $n \in \mathbb{N}$:

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Abelian tracial von Neumann algebras

$\Leftrightarrow$

Probability measure algebras.

Stability

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Pf. Immediate by the above lemma and the FHS characterization of stability. □

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Does CH imply that all tracial ultraproducts $\prod_{\mathcal{U}} M_n(\mathbb{C})$ *are isomorphic?*
I.e., do their theories converge?