The core model induction

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4. We will then explain some technicalities that arise in developing the necessary tools.
5. Warning: we will not have time to explain what a mouse is and what an iteration strategy is. We hope you learned this concepts from Schindler's tutorial and that you will gladly compute the 15th projectum if needed.

What is the core model induction?

It is a technique for calibrating lower bounds of consistency strengths of set theoretic statements.

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2. Combinatorial statements: $\neg\Box_\kappa$ where κ is a singular strong limit cardinal and etc.
3. Generic embeddings: generic embeddings given by precipitous ideals, dense ideals and etc.

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2. There is a collection of companion theorems that link the determinacy theories with large cardinal theories.
3. Both together give large cardinal lower bounds.

What kind of determinacy theories?

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2. A way of getting a hierarchy of axioms extending AD^+ is to consider Solovay sequence.

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2. If $\theta_\alpha < \Theta$ then $\theta_{\alpha+1} = \sup\{\alpha : \text{there is a surjection } f : \mathcal{P}(\theta_\alpha) \to \alpha$ such that f is ordinal definable $\}$,

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3. $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.
4. Ω is such that $\theta_\Omega = \Theta$.

The hierarchy: Solovay hierarchy

$$AD^+ + \Theta = \theta_0 <_{con} AD^+ + \Theta = \theta_1 <_{con} ...AD^+ + \Theta = \theta_\omega <_{con} ...AD^+ + \Theta = \theta_{\omega_1} <_{con} AD^+ + \Theta = \theta_{\omega_1+1} <_{con} \cdots$$

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$AD_{\mathbb{R}}$ + "Θ is regular" is a natural limit point of the hierarchy and is quite strong.

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3. (Woodin) If λ is a limit of Woodin cardinals and $< \lambda$-strong cardinals then the derived model at λ satisfies $\mathrm{AD}_{\mathbb{R}}$.

But where do these axioms hold?

Recall from Steel's talk,

$$A \in C^{\nu} \iff \exists F(F \text{ is a model operator on } H_{\nu} \text{ with parameter in } \mathbb{R}, \text{ and } A \text{ is definable over } \langle H_{\omega_1}, \in, F \restriction H_{\omega_1} \rangle).$$

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2. A certain K^c construction of $\mathrm{HOD}^{L(C^{\omega_2}, \mathbb{R})}$ is the model where it is shown that a certain large cardinal exists.

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1. Given a theory S from the Solovay hierarchy, is there $\Gamma \subseteq C^{\omega_2}$ such that $L(\Gamma, \mathbb{R}) \vDash S$?

To illustrate some of the technical ideas involved we concentrate on the following theorem.

Theorem (S.)

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1. $\mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) = \Gamma$,
2. $L(\Gamma, \mathbb{R}) \vDash$ *"there is no inner model M such that $\mathbb{R} \subseteq M$ and $M \vDash AD_{\mathbb{R}}$ + "Θ is regular""*.

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Then there is $A \subseteq \mathbb{R}$ such that $A \notin \Gamma$ and $L(A, \mathbb{R}) \vDash AD^+$.

Open Problem. It is not known how small ϵ is.

The theorem can be used to show that

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1. *Assume CH+"there is an ω_1-dense ideal on ω_1" $+ \epsilon$. Then there is $\Gamma \subseteq C^{\omega_2}$ such that $L(\Gamma, \mathbb{R}) \vDash AD_{\mathbb{R}}$ + "Θ is regular".*

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2. *Thus, Con(ZFC+CH+"there is an ω_1-dense ideal on ω_1" $+ \epsilon) \rightarrow$ Con($AD_{\mathbb{R}} +$ "Θ is regular").*

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2. *Thus, Con(ZFC+CH+"there is an ω_1-dense ideal on ω_1" $+ \epsilon) \to$ Con($AD_{\mathbb{R}}$ + "Θ is regular").*

Theorem (S.-Woodin)

The following theories are equiconsistent;

1. *ZFC+CH+"there is an ω_1-dense ideal on ω_1" $+ \epsilon$,*
2. *$AD_{\mathbb{R}}$ + "Θ is regular".*

So, fix Γ and $j : V \to M \subseteq V[G]$ as in the hypothesis. We are trying to construct a set of reals A such that $A \notin \Gamma$ and $L(A, \mathbb{R}) \vDash AD^+$. Where should we look for such an A?

Woodin's insight

Look for a countable "mouse" $\mathcal{M}$ such that $\mathcal{M}$ cannot have a strategy in Γ yet it has a strategy. Let A code the strategy of $\mathcal{M}$.

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Look for a countable "mouse" $\mathcal{M}$ such that $\mathcal{M}$ cannot have a strategy in Γ yet it has a strategy. Let A code the strategy of $\mathcal{M}$. But what should $\mathcal{M}$ be? How do we get it?

Woodin's other insight

Since in many situations we know that $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ is like a mouse, it is a *hybrid mouse* or rather *hod mouse*, show that it has a strategy and use this to get a strategy for something that is countable.

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Since in many situations we know that $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ is like a mouse, it is a *hybrid mouse* or rather *hod mouse*, show that it has a strategy and use this to get a strategy for something that is countable.
Plan: Get a strategy for HOD (which is not countable).

The final plan.

1. Let $\mathcal{H} = \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. Then in M, $\mathcal{H}$ is countable. Get a strategy for $\mathcal{H}$ in M.

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3. Let Σ be this strategy of $\mathcal{H}$ in M. Show that $j(\mathcal{H})$ is a Σ-iterate of $\mathcal{H}$.
4. By elementarity, there is $\mathcal{P}$ and Λ in V such that $\mathcal{H}$ is a Λ-iterate of $\mathcal{P}$.
5. Let A code Λ. Because $\mathcal{P}$ iterates to $\mathcal{H}$ via Λ, $A \notin \Gamma$. Show that $L(A,\mathbb{R}) \vDash AD^+$.

The missing step

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There are some guesses but nothing concrete.

The following theorem gives an upper bound, however.

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A consequence of this theorem is the following result;

Theorem (S.-Woodin)

It is consistent relative to a Woodin limit of Woodins that $MM^{+}(c)$ *holds.*

1. CH + "ω_1-dense ideal on ω_1" + ϵ + "$\omega-$presaturated ideal on ω_2" gives $\mathcal{M}_1$ of $AD_{\mathbb{R}}$ + "Θ is regular" (S.-Steel).

1. $CH +$ "ω_1-dense ideal on ω_1" $+ \epsilon +$ "$\omega-$presaturated ideal on ω_2" gives $\mathcal{M}_1$ of $AD_{\mathbb{R}} +$ "Θ is regular" (S.-Steel). It is an equiconsistency (Shelah-Woodin, for the other direction).

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2. $\neg\Box_\kappa$ where κ is a singular strong limit cardinal. Steel showed $AD^{L(\mathbb{R})}$. It seems to give a non-tame mouse when $\kappa > \aleph_\omega$ (S.-Schindler-Steel). A further work should give $AD_{\mathbb{R}}$ + "Θ is regular".

The skeptic's response to CMI

Theorem (JSSS, CMI Free)

PFA implies there exists a non-domestic mouse.

Getting back to propaganda, though

$AD_{\mathbb{R}}$ + "Θ is regular" is stronger than a non-domestic mouse.

What do we need to complete the plan?

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2. To show 1, one needs to prove the *Mouse Set Conjecture*.

Definition

The Mouse Capturing is the statement that for any two reals x and y, x is $OD(y)$ iff there is a mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$.

The Mouse Set Conjecture

Conjecture (Steel and Woodin)

Assume AD^+ and that there is no inner model with a superstrong cardinal. Then Mouse Capturing holds.

The Mouse Set Conjecture isn't wacky

Theorem

1. *(Kleene)* $x \in \Delta^1_1(y) \leftrightarrow x \in L_{\omega_1^{ck}(y)}[y]$.

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3. *etc.*

The hypo of Mouse Set Conjecture

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The hypo of Mouse Set Conjecture

1. Why AD^+? Mouse Capturing implies that $|\mathbb{R}^{HOD}| \leq \omega_1$ while it is consistent in ZFC+Large Cardinals that $V = HOD + \neg CH$.
2. Why no mouse with a superstrong? Because the notion of a mouse is well-defined and well-understood only below this large cardinal.

A partial result

Theorem (S.)

Assume AD^+ and there is no inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + “Θ is regular”. Then Mouse Capturing holds.

How are hods computed?

Assume Mouse Capturing and work under AD^+. As a first step, notice that if $x \in \mathrm{HOD}$ then x is in a mouse. So $\mathbb{R}^{\mathrm{HOD}}$ is a set of reals of a mouse. We just generalize this but it is much harder.

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HOD is shown to be a hod premouse.

Hybrid mice

Given a mouse $\mathcal{M}$ and an iteration strategy Σ for $\mathcal{M}$, one can construct mice with respect to Σ. These are called *hybrid mice* and have the form

$$L_\alpha[\vec{E}, \Sigma].$$

The hybrid mice we are interested in are the so-called *rigidly layered hybrid mice* or *"extender biased" hybrid mice*.

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draw a picture.

Hod mice

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Theorem (Woodin)

Assume AD$^+$. For every α, if $\theta_{\alpha+1}$ exists then it is a Woodin cardinal in HOD.

The hod theorems

Theorem (Woodin)

HOD *of the minimal model of* $AD_{\mathbb{R}}$ *is a hod premouse.*

The hod theorems

Theorem (S.)

HOD *of the minimal model of* $AD_{\mathbb{R}} +$ *"Θ is regular" is a hod premouse.*

This much is enough to carry out the plan.

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    3. in $\mathcal{P}$, the largest layer of $\mathcal{P}$ is a limit of Woodins and its cofinality in $\mathcal{P}$ is a measurable cardinal. In this case we usually say $\lambda^{\mathcal{P}}$ is limit or that we are in a hard limit case.

$B(\mathcal{P}, \Sigma)$ in the limit case

If $\mathcal{P}$ and $\mathcal{Q}$ are hod premice then

$$\mathcal{P} \trianglelefteq_{hod} \mathcal{Q} \text{ iff } \exists\alpha \leq \lambda^{\mathcal{Q}}(\mathcal{P} = \mathcal{Q}(\alpha))$$

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1. If $\mathcal{P}$ is a hod mouse and Σ is its strategy then we say $(\mathcal{P}, \Sigma)$ is a hod pair.

$B(\mathcal{P}, \Sigma)$ in the limit case

If $\mathcal{P}$ and $\mathcal{Q}$ are hod premice then

$$\mathcal{P} \trianglelefteq_{hod} \mathcal{Q} \text{ iff } \exists \alpha \leq \lambda^{\mathcal{Q}}(\mathcal{P} = \mathcal{Q}(\alpha))$$

1. If $\mathcal{P}$ is a hod mouse and Σ is its strategy then we say $(\mathcal{P}, \Sigma)$ is a hod pair.
2. If $\mathcal{T}$ is an iteration tree on $\mathcal{P}$ according to Σ with last model $\mathcal{Q}$ then $\Sigma_{\mathcal{Q},\mathcal{T}}$ is the corresponding tail of Σ.

$I(\mathcal{P}, \Sigma)$ and $B(P, \Sigma)$

1. If $(\mathcal{P}, \Sigma)$ is a hod pair then let

$$I(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \vec{\mathcal{T}}) : \vec{\mathcal{T}} \text{ is a stack on } \mathcal{P} \text{ according to } \Sigma\}.$$

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2. If $(\mathcal{P}, \Sigma)$ is a hod pair such that $\lambda^{\mathcal{P}}$ is limit then
$$B(\mathcal{P}, \Sigma) = \{(\mathcal{Q}, \vec{\mathcal{T}}) : \exists \mathcal{R}((\mathcal{R}, \vec{\mathcal{T}}) \in I(\mathcal{P}, \Sigma) \wedge \mathcal{Q} \triangleleft_{hod} \mathcal{R}\}.$$

$\Gamma(\mathcal{P}, \Sigma)$

Suppose $(\mathcal{P}, \Sigma)$ is hod pair such that $\lambda^{\mathcal{P}}$ is limit. Then let

$$\Gamma(\mathcal{P}, \Sigma) = \{A \subseteq \mathbb{R} : \exists(\mathcal{Q}, \vec{\mathcal{T}}) \in B(\mathcal{P}, \Sigma)(A \leq_w Code(\Sigma_{\mathcal{Q}, \vec{\mathcal{T}}})\}.$$

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Remark: $\Gamma(\mathcal{P}, \Sigma)$ can be defined even when $\lambda^{\mathcal{P}}$ is a successor but it is more technical.

Generation of full pointclasses

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Theorem (S.)

Assume there is no inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + "Θ is regular". Then Γ is a full pointclass iff either $\Gamma = \mathcal{P}(\mathbb{R})$ or $\Gamma = \Gamma(\mathcal{P}, \Sigma)$ for some hod pair $(\mathcal{P}, \Sigma)$.

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Theorem (S.)

Comparison is true for hod pairs below $AD_{\mathbb{R}}$ + "Θ is regular".

We are now ready for outlining the computation of HOD and showing how to prove the theorem we promissed. First lets deal with HOD.

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Let $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ be the direct limit of all iterates of $\mathcal{P}$ via Σ.

1. Because of comparison $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ is independent of $(\mathcal{P}, \Sigma)$ and depends only on α. This means that

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$$\mathcal{M}_\infty(\mathcal{P}, \Sigma) \subseteq \mathrm{HOD}.$$
2. Then one shows by a long induction that in fact
$$\mathcal{M}_\infty | \theta_\alpha = \mathrm{HOD} | \theta_\alpha.$$

The proof of the theorem

Theorem (S.)

Con(Woodin limit of Wodins) implies Con($AD_{\mathbb{R}}$ + "Θ is regular").

Divergent models of *AD*

1. We say there are divergent models of AD^+ if there are $A, B \subseteq \mathbb{R}$ such that $L(A, \mathbb{R}) \vDash AD^+$, $L(B, \mathbb{R}) \vDash AD^+$ but $L(A, B, \mathbb{R}) \vDash \neg AD^+$.

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2. If $A, B \subseteq \mathbb{R}$ form divergent models of AD^+ then let

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$$Com_{A,B} = \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \cap L(B, \mathbb{R}).$$
3. Woodin showed that $L(Com_{A,B}, \mathbb{R}) \vDash AD_{\mathbb{R}}$.

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Hence, it is enough to show that the existence of the divergent models of AD^+ gives a model of $AD_{\mathbb{R}}$ + "Θ is regular".

So, suppose $A, B \subseteq \mathbb{R}$ form divergent models of AD^+ yet there is no inner model of $AD_{\mathbb{R}}$ + "Θ is regular".

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2. Doing the same in $L(B, \mathbb{R})$, we get $(\mathcal{Q}, \Lambda) \in L(B, \mathbb{R})$ such that $Com_{A,B} = \Gamma(\mathcal{Q}, \Lambda)$.

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2. Doing the same in $L(B, \mathbb{R})$, we get $(\mathcal{Q}, \Lambda) \in L(B, \mathbb{R})$ such that $Com_{A,B} = \Gamma(\mathcal{Q}, \Lambda)$.
3. Notice that $(\mathcal{P}, \Sigma), (\mathcal{Q}, \Lambda) \notin Com_{A,B}$.

Now compare $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$. We get a common tail $(\mathcal{R}, \Psi)$.

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3. But then $(\mathcal{R}, \Psi) \in Com_{A,B}$.
4. However, $\Gamma(\mathcal{R}, \Psi) = \Gamma(\mathcal{P}, \Sigma) = \Gamma(\mathcal{Q}, \Lambda) = Com_{A,B}$, giving a contradiction.

The End.