Applied Proof Theory: Proof Interpretations and Their Use in Mathematics

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Corrected version Nov.20: a confused slide on the functional interpretation of weak compactness as well as a slide stating a bound on Browder's theorem have been deleted as the latter has been superseded meanwhile: weak compactess can be bypassed resulting in a primitive recursive bound.

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Lecture III: Applications in Nonlinear Analysis and Topological Dynamics.

Lecture I

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E.g. Let $C \equiv \forall x \in \mathbb{N} \exists y \in \mathbb{N}\, F(x, y)$

Naive Attempt: try to extract an explicit computable function realizing (or bounding) '$\exists y$': $\forall x \in \mathbb{N}\, F(x, f(x))$.

Naive attempt fails

Proposition

There exist a sentence $A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)$ in the language of arithmetic (A_{qf} quantifier-free and hence decidable), such

- A is logical valid,

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Proof: Take

$$A :\equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z) \big),$$

where T is the (primitive recursive) Kleene-T-predicate.

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Any bound g on '$\exists y$', i.e. no computable g such that

$$\forall x \exists y \leq g(x) \forall z\, (T(x,x,y) \vee \neg T(x,x,z))$$

since this would solve the halting problem! □

However, one can obtain such witness candidates and bounds (and even realizing function(al)s) for a weakened version A^H of A:

Definition

$A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)$. Then the Herbrand normal form of A is defined as

$$A^H :\equiv \exists x_1, x_2 A_{qf}(x_1, f(x_1), x_2, g(x_1, x_2)),$$

where f, g are new function symbols, called index functions.

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A and A^H are equivalent with respect to logical validity, i.e.

$$\models A \Leftrightarrow \models A^H,$$

but are not logically equivalent.

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allows to construct a list of candidates (uniformly in x, g) for '$\exists y$', namely $(c, g(c))$ (and also $(x, g(x))$) for any constant c):

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$$A^{H,D} :\equiv \big(P(x,c) \vee \neg P(x,g(c))\big) \vee \big(P(x,g(c)) \vee \neg P(x,g(g(c)))\big)$$

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is a tautology.

J. Herbrand's Theorem ('Théorème fondamental', 1930)

Theorem

Let $A \equiv \exists x_1 \forall y_1 \exists x_2 \forall y_2 A_{qf}(x_1, y_1, x_2, y_2)$. Then:
PL $\vdash A$ iff there are terms $s_1, \ldots, s_k, t_1, \ldots, t_n$ (built up out of the constants and variables of A and the **index functions** used for the formation of A^H) such that

$$A^{H,D} :\equiv \bigvee_{i=1}^{k} \bigvee_{j=1}^{n} A_{qf}(s_i, f(s_i), t_j, g(s_i, t_j))$$

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is a tautology. $A^{H,D}$ is called **Herbrand Disjunction**.
Note that the length of this disjunction is fixed: $k \cdot n$. The terms s_i, t_j can be extracted from a given PL-proof of A.

Herbrand's Theorem continued

Replacing in $A^{H,D}$ all terms '$g(s_i, t_j)$', '$f(s_i)$', by new variables (treating larger terms first) results in another tautological disjunction A^D s.t. A can be inferred from A by a **direct proof**.

Remark

- For sentences $A \equiv \forall x \exists y \forall z\, A_{qf}(x, y, z)$, A^D can be written in the form

$$A_{qf}(x, t_1, b_1) \vee A_{qf}(x, t_2, b_2) \vee \ldots \vee A_{qf}(x, t_k, b_k),$$

where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).

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  where the b_i are new variables and t_i does not contain any b_j with $i \leq j$ (used by Luckhardt's analysis of Roth's theorem, see below).

- Herbrand's theorem immediately extends to first-order theories $\mathcal{T}$ whose non-logical axioms $G_1, \ldots, G_n$ are all purely universal.

Theorem (Roth 1955)

An algebraic irrational number α has only finitely many exceptionally good rational approximations, i.e. for $\varepsilon > 0$ there are only finitely many $q \in \mathbb{N}$ such that

$$R(q) :\equiv q > 1 \land \exists! p \in \mathbb{Z} : (p, q) = 1 \land |\alpha - pq^{-1}| < q^{-2-\varepsilon}.$$

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Theorem (Luckhardt 1985/89)

The following upper bound on $\#\{q : R(q)\}$ holds:

$$\#\{q : R(q)\} < \frac{7}{3}\varepsilon^{-1} \log N_\alpha + 6 \cdot 10^3 \varepsilon^{-5} \log^2 d \cdot \log(50\varepsilon^{-2} \log d),$$

where $N_\alpha < \max(21 \log 2h(\alpha), 2\log(1 + |\alpha|))$ and h is the logarithmic absolute homogeneous height and $d = deg(\alpha)$.

Independently: Bombieri and van der Poorten 1988.

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- Techniques work only for restricted formal contexts: mainly purely universal ('algebraic') axioms, restricted use of induction, no higher analytical principles.
- Require that one can 'guess' the correct Herbrand terms: in general procedure results in proofs of length $2_n^{|P|}$, where $2_{n+1}^k = 2^{2_n^k}$ (n cut complexity).

Towards generalizations of Herbrand's theorem

Allow **functionals** $\Phi(x, f)$ instead of just Herbrand terms: Let's consider again the example

$$A \equiv \forall x \exists y \forall z \big(T(x, x, y) \vee \neg T(x, x, z) \big).$$

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A^H can be realized by a computable functional of type level 2 which is defined by cases:

$$\Phi(x, g) := \begin{cases} x & \text{if } \neg T(x, x, g(x)) \\ g(x) & \text{otherwise.} \end{cases}$$

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From this definition it easily follows that

$$\forall x, g \big(T(x, x, \Phi(x, g)) \vee \neg T(x, x, g(\Phi(x, g))) \big).$$

Φ satisfies **G. Kreisel's no-counterexample interpretation!**

If A is not provable in PL but e.g. in PA more **complicated functionals** are needed (Kreisel 1951):

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Let (a_n) be a nonincreasing sequence in $[0,1]$. Then, clearly, (a_n) is convergent and so a Cauchy sequence which we write as:

$$(1)\ \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \in \mathbb{N} \forall i,j \in [n;n+m]\ (|a_i - a_j| \leq 2^{-k}),$$

where $[n;n+m] := \{n, n+1, \ldots, n+m\}$.

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where $[n;n+m] := \{n, n+1, \ldots, n+m\}$.
Then the (partial) Herbrand normal form of this statement is

$$(2)\ \forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \in \mathbb{N} \forall i,j \in [n;n+g(n)]\ (|a_i - a_j| \le 2^{-k}).$$

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By contrast, there is a **simple (primitive recursive) bound** $\Phi^*(g,k)$ on (2) (also referred to as **'metastability'** by T.Tao):

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Proposition

Let (a_n) be any nonincreasing sequence in $[0,1]$ then

$$\forall k \in \mathbb{N} \forall g \in \mathbb{N}^{\mathbb{N}} \exists n \leq \Phi^*(g,k) \forall i,j \in [n; n+g(n)]\ (|a_i - a_j| \leq 2^{-k}),$$

where

$$\Phi^*(g,k) := \tilde{g}^{(2^k)}(0) \text{ with } \tilde{g}(n) := n + g(n).$$

Moreover, there exists an $i < 2^k$ such that n can be taken as $\tilde{g}^{(i)}(0)$.

Remark

The previous result can be viewed as a polished form of a **Herbrand disjunction** of **variable (in k) length**:

$$\bigvee_{i=0}^{2^k-1} \left(|a_{\tilde{g}^{(i)}(0)} - a_{\tilde{g}(\tilde{g}^{(i)}(0))}| \leq 2^{-k}\right).$$

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Corollary (T. Tao's finite convergence principle)

$$\forall k \in \mathbb{N}, g : \mathbb{N} \to \mathbb{N} \exists M \in \mathbb{N} \forall 1 \geq a_0 \geq \ldots \geq a_M \geq 0 \exists N \in \mathbb{N}$$
$$\big(N + g(N) \leq M \wedge \forall n, m \in [N, N + g(N)](|a_n - a_m| \ \leq 2^{-k})\big).$$

One may take $N := \Phi^*(g, k)$ as above.

An Example from Ergodic Theory

X **Hilbert space**, $f : X \to X$ **linear** and $\|f(x)\| \leq \|x\|$ for all $x \in X$.

$$A_n(x) := \frac{1}{n+1} S_n(x), \text{ where } S_n(x) := \sum_{i=0}^{n} f^i(x) \quad (n \geq 0).$$

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Theorem (Garrett Birkhoff 1939)

Mean Ergodic Theorem holds for uniformly convex Banach spaces.

Based on logical metatheorem to be discussed in the 2nd lecture:

Theorem (K./Leuştean, to appear in Ergodic Theor. Dynam. Syst.)

X uniformly convex Banach space, η a modulus of uniform convexity and $f: X \to X$ as above, $b > 0$.

Then for all $x \in X$ with $\|x\| \le b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \le \Phi(\varepsilon, g, b, \eta) \, \forall i, j \in [n; n + g(n)] \, \big(\|A_i(x) - A_j(x)\| < \varepsilon \big),$$

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where

$$\Phi(\varepsilon, g, b, \eta) := M \cdot \tilde{h}^K(0), \text{ with}$$

$$M := \left\lceil \frac{16b}{\varepsilon} \right\rceil, \gamma := \frac{\varepsilon}{16}\eta\left(\frac{\varepsilon}{8b}\right), \quad K := \left\lceil \frac{b}{\gamma} \right\rceil,$$

$$h, \tilde{h} : \mathbb{N} \to \mathbb{N},\ h(n) := 2(Mn + g(Mn)), \quad \tilde{h}(n) := \max_{i \le n} h(i).$$

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Special Hilbert case: treated prior by Avigad/Gerhardy/Towsner (again based on logical metatheorem).

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Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall n \in \mathbb{N} \forall f : \mathbb{N} \to C_n \exists i \leq n \forall k \in \mathbb{N} \exists m \geq k \left(f(m) = i \right).$$

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has trivial n.c.i.-solution for '$\exists i$','$\exists m$':

$$M(n, f, F) := \max\{F(i) : i \leq n\} \text{ and } I(n, f, F) := f(M(n, f, F)).$$

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Direct example: Infinitary Pigeonhole Principle (IPP):

$$\forall n \in \mathbb{N} \forall f : \mathbb{N} \to C_n \exists i \le n \forall k \in \mathbb{N} \exists m \ge k \left(f(m) = i \right).$$

IPP causes arbitrary **primitive recursive complexity**, but $(\mathrm{IPP})^H$

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$$M(n, f, F) := \max\{F(i) : i \le n\} \text{ and } I(n, f, F) := f(M(n, f, F)).$$

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Related problem: bad behavior w.r.t. modus ponens!

A Modular Approach: Proof Interpretations

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Our approach is based on novel forms and extensions of:

K. Gödel's functional interpretation!

Gödel's functional interpretation in five minutes

Gödel's **functional interpretation** D combined with Krivine's **negative translation** N results in an interpretation $Sh = D \circ N$ (Streicher/K.07)

$$A \mapsto A^{Sh} \quad \text{(Shoenfield variant)}$$

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- $A \leftrightarrow A^{Sh}$ by classical logic and **quantifier-free choice** in all types

$$\text{QF-AC}: \ \forall \underline{a} \exists \underline{b}\, F_{qf}(\underline{a}, \underline{b}) \to \exists \underline{B} \forall \underline{a}\, F_{qf}(\underline{a}, \underline{B}(\underline{a})).$$

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- $\underline{x}, \underline{y}$ are tuples of **functionals of finite type** over the base types of the system at hand,

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(Sh7) $(A \wedge B)^{Sh} \equiv \forall n, u, v \exists x, y\, (n{=}0 \to A_{Sh}(u,x)) \wedge (n{\neq}0 \to B_{Sh}(v,y))$
$$\leftrightarrow \forall u, v \exists x, y\, \big(A_{Sh}(u,x) \wedge B_{Sh}(v,y)\big).$$

Sh **extracts** from a given proof p

$$p \vdash \forall x \, \exists y \, A_{qf}(x, y)$$

an explicit effective functional Φ realizing A^{Sh}, i.e.

$$\forall x \, A_{qf}(x, \Phi(x)).$$

3. Monotone functional interpretation (K.1996)

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$$\exists Y \ \big(\Phi^* \gtrsim Y \wedge \forall x\, A_{Sh}(x, Y(x))\big),$$

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Monotone *Sh* extracts Φ^* such that

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where $\gtrsim$ is some suitable notion of being a 'bound' that applies to higher order function spaces (W.A. Howard)

$$\begin{cases} x^* \gtrsim_{\mathbb{N}} x :\equiv x^* \geq x, \\ x^* \gtrsim_{\rho\to\tau} x :\equiv \forall y^*, y (y^* \gtrsim_\rho y \to x^*(y^*) \gtrsim_\tau x(y)). \end{cases}$$

Also relevant: **bounded functional interpretation** (F. Ferreira, P. Oliva)

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The monotone functional interpretation of PCM coincides with the interpretation given above (Tao's finitary PCM).

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Full story in: Gaspar/K. 'On Tao's "finitary" infinite pigeonhole principle' (JSL, to appear).

Summarizing the discussion so far

- To exhibit the **finitary combinatorial/computational content (f.c.c.)** of ineffective principles P requires nontrivial transformations P^{MFI} of P as provided by **monotone functional interpretation (MFI)**. For $P \equiv \forall\exists$-sentence, P^{MFI} provides **uniform bound**.

Summarizing the discussion so far

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- **MFI** provides a **general method** for carrying out the extraction of this f.c.c. throughout a given proof all the way down to the conclusion (already for IPP this is nontrivial).

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- To exhibit the **finitary combinatorial/computational content (f.c.c.)** of ineffective principles P requires nontrivial transformations P^{MFI} of P as provided by **monotone functional interpretation (MFI)**. For $P \equiv \forall\exists$-sentence, P^{MFI} provides **uniform bound**.
- **MFI** provides a **general method** for carrying out the extraction of this f.c.c. throughout a given proof all the way down to the conclusion (already for IPP this is nontrivial).
- Specialized to the two prime examples of infinitary principles discussed in **Tao's essay 'Soft analysis, hard analysis and the finite convergence principle'**, MFI yields the finitary reformulations (with explicit bounds) suggested by Tao.

Tao on a finitary approach to analysis

'it is common to make a distinction between "hard", "quantitative", or "finitary" analysis on the one hand, and "soft", "qualitative", or "infinitary" analysis on the other hand.' ...'It is fairly well known that the results obtained by hard and soft analysis resp. can be connected to each other by various "correspondence principles" or "compactness principles". It is however my belief that the relationship between the two types of analysis is much deeper.' ...'There are rigorous results from proof theory which can allow one to automatically convert certain types of qualitative arguments into quantitative ones...'
(T. Tao: Soft analysis, hard analysis, and the finite convergence principle, 2007)

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1) Kohlenbach, U., Applied Proof Theory: Proof Interpretations and their Use in Mathematics. Springer Monographs in Mathematics. xx+536pp., Springer Heidelberg-Berlin, 2008.

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4) Tao, T., Soft analysis, hard analysis, and the finite convergence principle. In: 'Structure and Randomness. AMS, 298pp., 2008'.

5) Special issue of 'Dialectica' on Gödel's interpretation with contributions e.g. by Ferreira, Kohlenbach, Oliva, 2008.

Lecture II

General logical metatheorems I

- Context: **continuous functions** between constructively represented **Polish spaces**.

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- Uniformity w.r.t. parameters from **compact** Polish spaces.
- Extraction of **bounds** from **ineffective** existence proofs.

K., 1993-96: P Polish space, K a compact P-space, $A_\exists$ existential.
BA:= basic arithmetic, HBC Heine/Borel compactness (SEQ^- restricted sequential compactness).

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Important:

$\Phi(f_x)$ does not depend on $y \in K$ but on a representation f_x of x!

Logical comments

- Heine-Borel compactness = WKL (binary König's lemma).
  WKL $\vdash$ strict-$\Sigma^1_1 \leftrightarrow \Pi^0_1$
  (see applications in algebra by Coquand, Lombardi, Roy ...)

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  WKL $\vdash$ strict-$\Sigma^1_1 \leftrightarrow \Pi^0_1$
  (see applications in algebra by Coquand, Lombardi, Roy ...)
- Restricted **sequential compactness** = **restricted arithmetical comprehension.**

Limits of Metatheorem for concrete spaces

Compactness means constructively: **completeness** and **total boundedness**.

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Compactness means constructively: **completeness** and **total boundedness**.

Necessity of completeness: The set $[0,2]_{\mathbb{Q}}$ is totally bounded and constructively representable and

$$\mathrm{BA} \vdash \forall q \in [0,2]_{\mathbb{Q}} \; \exists n \in \mathbb{N}(|q - \sqrt{2}| >_{\mathbb{R}} 2^{-n}).$$

However: **no uniform bound on $\exists n \in \mathbb{N}$!**

Necessity of total boundedness: Let B be the unit ball $C[0,1]$. B is bounded and constructively representable.
By Weierstraß' theorem

$$\mathrm{BA} \vdash \forall f \in B \exists n \in \mathbb{N} \big(n \text{ code of } p \in \mathbb{Q}[X] \text{ s.t. } \|p - f\|_\infty < \frac{1}{2} \big)$$

but **no uniform bound** on $\exists n$: take $f_n := \sin(nx)$.

Necessity of $A_\exists$ '∃-formula':

Let (f_n) be the usual sequence of spike-functions in $C[0,1]$, s.t. (f_n) converges pointwise but not uniformly towards 0. Then

$$\mathrm{BA} \vdash \forall x \in [0,1] \forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \in \mathbb{N} (|f_{n+m}(x)| \leq 2^{-k}),$$

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but **no uniform bound** on '$\exists n$' (proof based on Σ^0_1-LEM).

Uniform bound only if $(f_n(x))$ **monotone** (Dini): **'$\forall m \in \mathbb{N}$' superfluous!**

Necessity of $\Phi(f_x)$ depending on a representative of x :

Consider

$$\text{BA} \vdash \forall x \in \mathbb{R} \exists n \in \mathbb{N}((n)_{\mathbb{R}} >_{\mathbb{R}} x).$$

Suppose there would exist an $=_{\mathbb{R}}$-extensional computable $\Phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ producing such a n. Then Φ would represent a **continuous** and hence **constant** function $\mathbb{R} \to \mathbb{N}$ which gives a contradiction.

Unique existence

X, K Polish, K compact, $f : X \times K \to \mathbb{R}$ (BA-definable).

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into **moduli of uniqueness** $\Phi : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$

$$\forall x \in X, y_1, y_2 \in K, \varepsilon > 0 \big(\bigwedge_{i=1}^{2} |f(x, y_i)| < \Phi(x, \varepsilon) \to d_K(y_1, y_2) < \varepsilon \big).$$

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$$d_K(\widehat{y}, y_{\Phi(x,\varepsilon)}) < \varepsilon).$$

Case study: strong unicity in L_1-approximation

P_n space of polynomials of degree $\leq n$, $f \in C[0,1]$,

$\|f\|_1 := \int_0^1 |f|, \quad dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1.$

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Best **approximation in the mean** of $f \in C[0,1]$:

$$\forall f \in C[0,1] \exists! p_b \in P_n \big(\|f - p_b\|_1 = dist_1(f, P_n)\big)$$

(existence **and** uniqueness use: WKL!)

Theorem (K./Paulo Oliva, APAL 2003)

Let $dist_1(f, P_n) := \inf_{p \in P_n} \|f - p\|_1$ and ω a modulus of uniform continuity for f.

$$\Psi(\omega, n, \varepsilon) := \min\{\frac{c_n \varepsilon}{8(n+1)^2}, \frac{c_n \varepsilon}{2} \omega_n(\frac{c_n \varepsilon}{2})\}, \textit{ where}$$
$$c_n := \frac{\lfloor n/2 \rfloor ! \lceil n/2 \rceil !}{2^{4n+3}(n+1)^{3n+1}} \textit{ and}$$
$$\omega_n(\varepsilon) := \min\{\omega(\frac{\varepsilon}{4}), \frac{\varepsilon}{40(n+1)^4 \lceil \frac{1}{\omega(1)} \rceil}\}.$$

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Then $\forall n \in \mathbb{N}, p_1, p_2 \in P_n$

$$\forall \varepsilon \in \mathbb{Q}^*_+ \Big(\bigwedge_{i=1}^{2} (\|f - p_i\|_1 - dist_1(f, P_n) \le \Psi(\omega, n, \varepsilon)) \to \|p_1 - p_2\|_1 \le \varepsilon \Big).$$

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- Ψ provides the **first effective version** of results due to Bjoernestal (1975) and Kroó (1978-1981).
- Kroó (1978) implies that the ε-dependency in Ψ is **optimal**.
- Ψ allows the **first complexity upper bound** for the sequence of best L_1-approximations (p_n) in P_n of poly-time functions $f \in C[0,1]$ (P. Oliva, MLQ 2003).

The nonseparable/noncompact case

Proposition

Let $(X, \|\cdot\|)$ be a strictly convex normed space and $C \subseteq X$ a convex subset. Then any point $x \in X$ has at most one point $c \in C$ of minimal distance, i.e. $\|x - c\| = \text{dist}(x, C)$.

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Hence: if X is separable and complete and provably strictly convex and C compact, then one can extract a modulus of uniqueness.

Observation: compactness only used to extract uniform bound on strict convexity ($=$ **modulus of uniform convexity**) from proof of strict convexity.

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$$\Phi(\varepsilon) := \min\left(1, \frac{\varepsilon}{4}, \frac{\varepsilon}{4} \cdot \frac{\eta(\varepsilon/(d+1))}{1-\eta(\varepsilon/(d+1))}\right).$$

Conclusion: neither compactness nor separability required!

General logical metatheorems II

Many abstract types of metric structures can be added as atoms: metric, hyperbolic, CAT(0), δ-hyperbolic, normed, uniformly convex, Hilbert, ... spaces or $\mathbb{R}$-trees X : add **new base type** X, all **finite types over** $\mathbb{N}, X$ and a new **constant** d_X representing d etc.

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Counterexamples (to extractibility of uniform bounds): for the classes of strictly convex ($\rightarrow$ uniformly convex) or separable ($\rightarrow$ totally bounded) spaces!

A formal system for analysis

Types: (i) $\mathbb{N}, X$ are types, (ii) with ρ, τ also $\rho \to \tau$ is a type.

Functionals of type $\rho \to \tau$ map type-ρ objects to type-τ objects.

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$\mathcal{A}^{\omega}[X, d, \ldots]$ results by adding constants $d_X, \ldots$ with axioms expressing that $(X, d, \ldots)$ is a nonempty metric, hyperbolic ... space.

A warning concerning equality

Extensionality rule (**only!**):

$$\frac{s =_\rho t}{r(s) =_\tau r(t)},$$

where only $x =_{\mathbb{N}} y$ primitive equality predicate but for $\rho \to \tau$

$$s^X =_X t^X :\equiv d_X(x, y) =_{\mathbb{R}} 0_{\mathbb{R}},$$
$$s =_{\rho\to\tau} t :\equiv \forall v^\rho (s(v) =_\tau t(v)).$$

A novel form of majorization

y, x functionals of types $\rho, \widehat{\rho} := \rho[\mathbb{N}/X]$ and a^X of type X:

$$x^{\mathbb{N}} \gtrsim^a_{\mathbb{N}} y^{\mathbb{N}} :\equiv x \geq y$$
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Normed linear case: $a := 0_X$.

Hyperbolic spaces

Definition (Takahashi,Kirk,Reich)

A **hyperbolic space** is a triple (X, d, W) where (X, d) is metric space and $W : X \times X \times [0, 1] \to X$ s.t.

(i) $d(z, W(x, y, \lambda)) \leq (1 - \lambda)d(z, x) + \lambda d(z, y)$,

(ii) $d(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot d(x, y)$,

(iii) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$,

(iv) $d(W(x, z, \lambda), W(y, w, \lambda)) \leq (1 - \lambda)d(x, y) + \lambda d(z, w)$.

- **CAT(0)-spaces (Gromov)** are hyperbolic spaces (X, d, W) which satisfy the **CN**-inequality of Bruhat-Tits

$$\begin{cases} d(y_0, y_1) = \frac{1}{2}d(y_1, y_2) = d(y_0, y_2) \rightarrow \\ d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2. \end{cases}$$

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Small types (over $\mathbb{N}, X$): $\mathbb{N}$, $\mathbb{N} \rightarrow \mathbb{N}$, X, $\mathbb{N} \rightarrow X$, $X \rightarrow X$.

Theorem (Gerhardy/K., Trans.Amer.Math.Soc. 2008)

Let P, K be Polish resp. compact metric spaces, $A_\exists$ $\exists$-formula, $\underline{\tau}$ small. If $\mathcal{A}^\omega[X,d,W]$ **proves**

$$\forall x \in P \forall y \in K \forall \underline{z}^{\underline{\tau}} \exists v^{\mathbb{N}} A_\exists(x,y,\underline{z},v),$$

then one can extract a **computable** $\Phi : \mathbb{N}^{\mathbb{N}} \times \underline{\mathbb{N}}^{(\mathbb{N})} \to \mathbb{N}$ s.t. the following holds in every nonempty hyperbolic space: for all representatives $r_x \in \mathbb{N}^{\mathbb{N}}$ of $x \in P$ and all $\underline{z}^{\underline{\tau}}$ and $\underline{z}^* \in \mathbb{N}^{(\mathbb{N})}$ s.t. $\exists a \in X(\underline{z}^* \gtrsim^a_{\underline{\tau}} \underline{z})$:

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For the bounded cases: K. Trans.AMS 2005.

As special case of **general logical metatheorems** due to Gerhardy/K. (Trans. Amer. Math. Soc. 2008) one has:

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If $\mathcal{A}^{\omega}[X, d, W]$ proves

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Normed case: also $\|z\| \leq b$.

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Since Birkhoff's proof formalizes in $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ the following is guaranteed:

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Since Birkhoff's proof formalizes in $\mathcal{A}^\omega[X, \|\cdot\|, \eta]$ the following is guaranteed:

X uniformly convex Banach space with modulus η and $f : X \to X$ nonexpansive linear operator. Let $b > 0$. Then there is an effective functional Φ in ε, g, b, η s.t. for all $x \in X$ with $\|x\| \leq b$, all $\varepsilon > 0$, all $g : \mathbb{N} \to \mathbb{N}$:

$$\exists n \leq \Phi(\varepsilon, g, b, \eta)\, \forall i, j \in [n, n + g(n)] \left(\|A_i(x) - A_j(x)\| < \varepsilon\right).$$

(see Lecture I)

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‘We shall establish Theorem 1.6 by “finitary ergodic theory” techniques, reminiscent of those used in [Green-Tao]...’ ‘The main advantage of working in the finitary setting ... is that the underlying dynamical system becomes extremely explicit’...‘In proof theory, this finitisation is known as Gödel functional interpretation...which is also closely related to the Kreisel no-counterexample interpretation’
(T. Tao: Norm convergence of multiple ergodic averages for commuting transformations, Ergodic Theor. and Dynam. Syst. 28, 2008)

Projections and Weak Compactness without separability

(to appear in: Festschrift for G. Mints)

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- the **weak compactness of $B_1(0)$** (here only countable choice for arithmetical formulas needed and restricted induction).

A theorem of F.E. Browder

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Corollary by Metatheorem: There is a functional $\Phi(k, g)$ (definable by primitive recursion and bar recursion of lowest type) such that

$$\forall k \in \mathbb{N} \, \forall g : \mathbb{N} \to \mathbb{N} \, \exists n \leq \Phi(k, g) \, \forall i, j \in [n; n + g(n)] \, (\|u_i - u_j\| < 2^{-k}).$$

Note that Φ does not depend on U, v_0 or X!

Lecture III

Applications to metric fixed point theory

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Theorem (Ishikawa 1976, Goebel/Kirk 1983)

(Ishikawa I)

If (x_n) is bounded, then $d(x_n, f(x_n)) \to 0$.

Logical analysis of the proof of Ishikawa's theorem

Let $K \in \mathbb{N}$ and $\alpha : \mathbb{N} \to \mathbb{N}$ be such that

$$(\lambda_n)_{n\in\mathbb{N}} \in [0, 1 - \frac{1}{K}]^{\mathbb{N}} \text{ and } \forall n \in \mathbb{N}(n \leq \sum_{i=0}^{\alpha(n)} \lambda_i).$$

Logical metatheorem applied to proof of Ishikawa's theorem yields computable Ψ, Φ s.t. for all $k \in \mathbb{N}$ and n.e. f

$$\forall i, j \leq \Psi(K, \alpha, b, \tilde{b}, k) \left(d(x, f(x)) \leq b \wedge d(x_i, x_j) \leq \tilde{b}\right) \to$$
$$\forall m \geq \Phi(K, \alpha, b, \tilde{b}, k) \left(d(x_m, f(x_m)) < 2^{-k}\right).$$

holds in **any (nonempty) hyperbolic space** (X, d, W).

Theorem (K.2007, K./Leustean AAA 2003)

$(X, d, W), (\lambda_n), K, \alpha$ as above, $f : X \to X$ nonexpansive the following holds for all $\varepsilon, b, \tilde{b} > 0$:

If $d(x, f(x)) \le b$ and $\forall i \le \Phi \forall j \le \alpha(\Phi, M)\ (d(x_i, x_{i+j}) \le \tilde{b})$ then $\forall n \ge \Phi\ \big(d(x_n, f(x_n)) \le \varepsilon\big)$,

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where

$$\Phi := \Phi(K, \alpha, b, \tilde{b}, \varepsilon) := \widehat{\alpha}\left(\left\lceil \frac{\tilde{b}\cdot \exp\left(K\cdot\left(\frac{\tilde{b}+3b}{\varepsilon}+1\right)\right)}{\varepsilon}\right\rceil \dot{-} 1, M\right),$$
$$M := \left\lceil \frac{\tilde{b}+3b}{\varepsilon} \right\rceil,$$
$$\widehat{\alpha}(0, n) := \tilde{\alpha}(0, n),\ \ \widehat{\alpha}(i+1, n) := \tilde{\alpha}(\widehat{\alpha}(i, n), n) \text{ with}$$
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$$\tilde{\alpha}(i, n) := i + \alpha(i, n)\ \ (i, n \in \mathbb{N})$$

with α s.t.

$$\forall i, n \in \mathbb{N}\big((\alpha(i, n) \le \alpha(i+1, n)) \wedge (n \le \sum_{s=i}^{i+\alpha(i,n)-1} \lambda_s)\big).$$

Known uniformity results in the bounded case

blue = hyperbolic, **green** = dir.nonex., **red** = both.

- Krasnoselski(1955): X unif.convex, C compact, $\lambda_k = \frac{1}{2}$, no uniform.
- Browder/Petryshyn(1967): X unif.convex, $\lambda_k = \lambda$, no uniformity.
- Groetsch(1972): X unif. convex, general λ_k, X, no uniformity
- Ishikawa (1976): No uniformity
- Edelstein/O'Brien (1978): Uniformity w.r.t. $x_0 \in C$ $(\lambda_k := \lambda)$
- Goebel/Kirk (1982): Uniformity w.r.t. x_0 **and** f. General λ_k
- Kirk/Martinez (1990): Uniformity for unif. convex $X, \lambda := 1/2$
- Goebel/Kirk (1990): Conjecture: no uniformity w.r.t. C
- Baillon/Bruck (1996): Uniformity w.r.t. x_0, f, C for $\lambda_k := \lambda$
- **Kirk (2001): Uniformity w.r.t. x_0, f for constant λ**
- Kohlenbach (2001): Full uniformity for general λ_k
- **K./Leustean (2003): Full uniformity for general λ_k**

Corollary (K.2007)

(generalizes result by Baillon-Bruck-Reich from 1978) Let (λ_n) *in* $[a, b] \subset (0, 1)$.

$$\textit{If } \lim_{n\to\infty} \frac{c(n)}{n} \to 0, \textit{ where } c(n) := \max\{d(x, x_j) : j \leq n\},$$

then

$$\lim_{n\to\infty} d(x_n, f(x_n)) = 0.$$

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Result optimal: $c(n) \le K \cdot n$ not sufficient!

Theorem (Ishikawa, Goebel, Kirk)

(Ishikawa II) If previous assumptions and X **compact**, then (x_n) converges towards a fixed point.

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Proof: Since X is compact, (x_n) possesses a **convergent subsequence** (x_{n_k}). Let $\widehat{x} := \lim x_{n_k}$. Since by Ishikawa I, (x_n) (and hence x_{n_k}) is an asymptotic fixed point sequence and f is continuous, $\widehat{x}$ is a fixed point of f. The claim now follows from the following easy inequality

$$\forall u \in Fix(f) \forall n \in \mathbb{N} \, (d(x_{n+1}, u) \leq d(x_n, u)).$$

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Problem: No computable rate of convergence.
Cauchy property $\forall\exists\forall$ rather than $\forall\exists$ (asymptotic regularity).
Best possible: Bound on the **no-counterexample interpretation**:

$$\text{(H) } \forall g : \mathbb{N} \to \mathbb{N} \forall k \exists n \forall j_1, j_2 \in [n; n + g(n)](d(x_{j_1}, x_{j_2}) < 2^{-k}).$$

Logical Metatheorem for Compact Spaces

We add to $\mathcal{T}[X, d, W]$ compactness via

- A constant $\gamma : \mathbb{N} \to \mathbb{N}$ with an axiom expressing that γ is a **modulus of total boundedness.**

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The completeness issue is of minor relevance for the case at hand, but the total boundedness is.

Two ways of expressing total boundedness

Definition

(first form): Add constants $\gamma^{\mathbb{N}\to\mathbb{N}}, a_{(\cdot)}^{\mathbb{N}\to X}$ with the universal axiom

$$(\text{TOT I}): \ \forall k \in \mathbb{N}, x \in X \exists n \le \gamma(k)(d(x, a_n) \le 2^{-k}).$$

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Definition

(second form): Add only constant $\gamma^{\mathbb{N}\to\mathbb{N}}$ with the universal axiom

$$(\text{TOT II}): \ \forall k \in \mathbb{N}, x_{(\cdot)}^{\mathbb{N}\to X} \exists i, j (i < j \leq \gamma(k)(d(x_i, x_j) \leq 2^{-k}).$$

Corresponding theories $\mathcal{T}[X, d, W, \mathcal{C}, \text{TOT I}]$ and $\mathcal{T}[X, d, W, \mathcal{C}, \text{TOT II}]$

Discussion

- Metatheorems for $\mathcal{T}[X, d, W, \mathcal{C}, \text{TOT I}]$ produce bounds depending on majorants for γ $a_{(\cdot)}$. For γ just take $\gamma^M(n) := \max\{\gamma(i) : i \leq n\}$ majorizes γ.

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- Benefits of $\mathcal{T}[X, d, W, \mathcal{C}, \text{TOT II}]$: **greater uniformity** of the bound.

Guaranteed by logical metatheorem

From the fact that the proof of

$$\text{Ishikawa I}(x_n) \ \wedge \ \text{BW}(x_n) \rightarrow \ \text{Ishikawa II}(x_n)$$

can be formalized in an appropriate fragment of $\mathcal{A}^\omega[X, d, W, \mathcal{C}, \text{TOT II}]$ it follows:

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Theorem

There exists a **primitive recursive functional** Ψ such that for any **rate of asymptotic regularity** Φ and any **modulus of total boundedness** γ for C, any g, k :

$$\exists n \leq \Psi(\Phi, \gamma, g, k) \forall j_1, j_2 \in [n; n + g(n)](d(x_{j_1}, x_{j_2}) < 2^{-k}).$$

Theorem (K., Nonlinear Analysis 2005)

A bound satisfying the previous theorem is given by

$$\Psi(\Phi,\gamma,g,k) := \max_{i\le\gamma(k)} \Psi_0(i,k,g,\Phi),$$

where

$$\begin{cases} \Psi_0(0,k,g,\Phi) := 0 \\ \Psi_0(n+1,k,g,\Phi) := \Phi\left(2^{-k-2}/(\max_{i\le n} g(\Psi_0(i,k,g,\Phi))+1)\right). \end{cases}$$

Asymptotically nonexpansive mappings

Let (X, d, W) be a hyperbolic space.

Definition (Goebel/Kirk,1972)

$f : X \to X$ is said to be **asymptotically nonexpansive with sequence** $(k_n) \in [0, \infty)^{\mathbb{N}}$ if $\lim_{n \to \infty} k_n = 0$ and

$$d(f^n(x), f^n(y)) \leq (1 + k_n) d(x, y), \ \forall n \in \mathbb{N}, \forall x, y \in X.$$

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$$x_0 := x \in X, \ x_{n+1} := (1 - \lambda_n)x_n \oplus \lambda_n f^n(x_n).$$

Theorem (Rhoades,Schu,Qihou,K./Lambov(2004),K./Leustean(2007))

Let (X, d, W) uniformly convex hyperbolic space and $(k_n) \subset \mathbb{R}_+$ with $\sum k_n < \infty$. Let $k \in \mathbb{N}$ and $\lambda_n \in [a, b]$ with $0 < a < b < 1$. $f : X \to X$ asymptotically weakly nonexpansive.

If f possesses a fixed point, then $d(x_n, f(x_n)) \stackrel{n\to\infty}{\to} 0$.

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Proof uses sequential compactness in the form of

Lemma

Let $(a_n),(b_n),(c_n)$ be sequences in $\mathbb{R}_+$ with

$$a_{n+1} \leq (1+b_n)a_n + c_n \quad (n \in \mathbb{N})$$

with $\sum b_n < \infty$, $\sum c_n < \infty$. Then (a_n) is convergent.

Theorem (K./Leuştean, to appear in: JEMS)

(X,d,W) uniformly convex with modulus η. $f: X \to X$ asymptotically n.e. with sequence (k_n). $\sum_{n=0}^{\infty} k_n \le K \in \mathbb{N}$ and $L \in \mathbb{N}, L \ge 2$ s.t. $\frac{1}{L} \le \lambda_n \le 1 - \frac{1}{L}$ for all $n \in \mathbb{N}$.

Let $x \in X$ and $b > 0$ be such that for any $\delta > 0$ there is $p \in X$ with

$$d(x,p) \le b \wedge d(f(p),p) \le \delta.$$

Then for all $\varepsilon \in (0,1]$ and for all $g : \mathbb{N} \to \mathbb{N}$,

$$\exists N \le \Phi(K,L,b,\eta,\varepsilon,g) \forall m \in [N, N+g(N)] \left(d(x_m, f(x_m)) < \varepsilon \right),$$

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where $\Phi(K,L,b,\eta,\varepsilon,g):=h^{(M)}(0),\quad h(n):=g(n+1)+n+2,$

$$M:=\left\lceil \frac{3\left(5KD+D+\frac{11}{2}\right)}{\delta}\right\rceil,\quad D:=e^K\,(b+2)\,,$$

$$\delta:=\frac{\varepsilon}{L^2F(K)}\cdot\eta\left((1+K)D+1,\frac{\varepsilon}{F(K)((1+K)D+1)}\right),$$

$$F(K):=2(1+(1+K)^2(2+K)).$$

Kirk's theorem for asymptotic contractions

Definition (Kirk JMAA03)

(X, d) metric space. $f : X \to X$ is an **asymptotic contraction** with moduli $\Phi, \Phi_n : [0, \infty) \to [0, \infty)$ if Φ, Φ_n are continuous, $\Phi(s) < s$ for all $s > 0$ and

$$\forall n \in \mathbb{N} \forall x, y \in X(d(f^n(x), f^n(y)) \leq \Phi_n(d(x, y)),$$

and $\Phi_n \to \Phi$ uniformly on the range of d.

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Theorem (Kirk JMAA03)

(X, d) complete metric space, $f : X \to X$ continuous asymptotic contraction with some orbit bounded. Then f has a unique fixed point $p \in X$ and $(f^n(x_0))$ converges to p for each $x_0 \in X$.

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(Proof uses ultrapower structures!)

- By proof mining P. Gerhardy (JMAA 2006, communicated by Kirk) obtained an **effective rate of proximity** Φ in appropriate moduli with elementary proof such that for the fixed point p

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- E.M.Briseid showed that for bounded metric spaces the existence of a x_0-uniform rate of convergence **implies** that f is asymptotically contractive (JMAA 2007). Also: new uniformity results generalizing Reich et al (2007).

Generalized p-contractive mappings

Definition: [Rhoades 1977] (X, d) metric space and $p \in \mathbb{N}$.
$f : X \to X$ is called **generalized p-contractive** if

$$\forall x, y \in X \big(x \neq y \to d(f^p(x), f^p(y)) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}\big).$$

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Theorem: [Kincses/Totik 1990]
(K, d) **compact** metric space and $f : K \to K$ continuous and generalized p-contractive for some $p \in \mathbb{N}$. Then f has a unique fixed point ξ and for every $x \in K$ $\lim_{n\to\infty} f^n(x) = \xi$.

Definition: [Briseid, J. Nonlinear Convex Anal. 2008]
(X,d) metric space, $p \in \mathbb{N}$. $f: X \to X$ is called **uniformly generalized p-contractive** with modulus $\eta : \mathbb{Q}^*_+ \to \mathbb{Q}^*_+$ if for all $x, y \in, \varepsilon \in \mathbb{Q}^*_+$

$$d(x,y) > \varepsilon \to d(f^p(x), f^p(y)) + \eta(\varepsilon) < \text{ diam}\,\{x, y, f^p(x), f^p(y)\}.$$

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Theorem: [Briseid, J. Nonlinear Convex Anal. 2008] (X,d) **complete** metric space and $p \in \mathbb{N}$. $f : X \to X$ be a uniformly continuous and uniformly generalized p-contractive with moduli ω, η. Let $(f^n(x_0))$ be bounded by $b \in \mathbb{Q}^*_+$. Then f has a unique fixed point ξ and $(f^n(x_0))$ converges to ξ with rate of convergence $\Phi : \mathbb{Q}^*_+ \to \mathbb{N}$,

$$\Phi(\varepsilon) := \begin{cases} p\lceil (b-\varepsilon)/\rho(\varepsilon)\rceil \text{ if } b > \varepsilon, \\ 0, \text{otherwise} \end{cases}$$

with

$$\rho(\varepsilon) := \min\left\{\eta(\varepsilon), \frac{\varepsilon}{2}, \eta(\frac{1}{2}\omega^p(\frac{\varepsilon}{2}))\right\}.$$

Applications in Topological Dynamics

Theorem (Multiple Birkhoff Recurrence)

Let (X, d) be a compact metric space and $T_1, \ldots, T_k$ commuting homeomorphisms of X. Then there exists $x \in X$ s.t.

$$\forall \varepsilon > 0 \exists n > 0 \bigwedge_{i=1}^{k} d(T_i^n(x), x) \leq \varepsilon.$$

Theorem (Gerhardy, Notre Dame J. of Formal Logic 2008)

Let γ be a modulus of total boundedness of (X, d), $T_1, \ldots, T_k$ commuting homeomorphisms of X with common modulus of uniform continuity ω and G the group generated from the T_i. Then

$$\forall \varepsilon > 0 \exists N, M > 0 \bigwedge_{i=1}^{k} \min_{n \leq N} \min_{g \in G_M} d(T_i^n(gx), gx) < \varepsilon,$$

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$$\begin{aligned}
&N=N^k(\varepsilon,\gamma,\omega),\ \ M=M^k(\varepsilon,\gamma,\omega),\\
&N^1(\varepsilon,\gamma,\omega)=M^1(\varepsilon,\gamma,\omega)=\gamma(\varepsilon/2)\\
&N^{m+1}(\varepsilon,\gamma,\omega)=\Phi_N^{m+1}(\gamma(\varepsilon/2))\cdot\gamma(\varepsilon/2),\\
&M^{m+1}(\varepsilon,\gamma,\omega)=\Phi_M^{m+1}(\gamma(\varepsilon/2))\cdot\gamma(\varepsilon/2),\\
&\Phi_N^{m+1}(i)=N^m(\varepsilon_i^{m+1},\gamma,\omega^2),\\
&\Phi_M^{m+1}(i)=2M^m(\varepsilon_i^{m+1},\gamma,\omega^2)+N^m(\varepsilon_i^{m+1},\gamma,\omega^2),\\
&\varepsilon_1^{m+1}=\varepsilon/4,\ \ \varepsilon_{i+1}^{m+1}=\omega^{\Phi_N^{m+1}(i)+\Phi_M^{m+1}(i)}(\varepsilon_i^{m+1}/2).
\end{aligned}$$

Further applications of proof theory to mathematics

- Numerous further applications in metric fixed point theory (Briseid, Gerhardy, Lambov, Leustean, K.) published in: J.Math.Anal.Appl.(4), Nonlinear Analysis (2), Numer.Funct.Anal.Opt.(2), Trans AMS (2), J. EMS (1), Convex Analysis (1), Abstr.Appl.Anal.(1), Fixed Point Theory (1), Proc. Fixed Point Theory (1).
- New results on Hilbert's 17th problem (Delzell, Inventiones Math. etc.)
- Proof theory and Ramsey's theorem for pairs: see the talk be A. Kreuzer at this meeting (Wednesday 14.00).

U. KOHLENBACH

Applied Proof Theory:
Proof Interpretations and their Use in Mathematics

Ulrich Kohlenbach presents an applied form of proof theory that has led in recent years to new results in number theory, approximation theory, nonlinear analysis, geodesic geometry and ergodic theory (among others). This applied approach is based on logical transformations (so-called proof interpretations) and concerns the extraction of effective data (such as bounds) from *prima facie* ineffective proofs as well as new qualitative results such as independence of solutions from certain parameters, generalizations of proofs by elimination of premises.
The book first develops the necessary logical machinery emphasizing novel forms of Gödel's famous functional (,Dialectica') interpretation. It then establishes general logical metatheorems that connect these techniques with concrete mathematics. Finally, two extended case studies (one in approximation theory and one in fixed point theory) show in detail how this machinery can be applied to concrete proofs in different areas of mathematics.

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