Dependence Logic

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The dependence concept

Dependence of health on genes.

Dependence of future events on past decisions.

Dependence of moves of a player on previous moves.

Arrow’s Theorem

If the social welfare function respects unanimity and independence of irrelevant alternatives, it is a dictatorship.

Question

Can one add the *dependence* concept to
first order logic (or other logics) in a coherent way?

What is the *logic* of dependence?

Solution

- We consider the strongest form a dependence, namely functional determination $z = f(x_1,\ldots,x_n)$, where $x_1,\ldots,x_n,z$ are individual variables.

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- We denote it $=\!(x_1,\ldots,x_n,z)$ and call it a dependence atom.
- In computer science: $x_1\ldots x_n \Rightarrow z$, where $x_1,\ldots,x_n,z$ are database fields. (Armstrong relation)

	Name	Job	Gender	Salary group
s_0	Jeff	analyst	M	C
s_1	Paula	assistant	F	A
s_2	Laurie	assistant	M	C

Multitude

- Dependence does not manifest itself in a **single** play, event or observation.
- The underlying concept of dependence logic is a multitude – a **collection** - of such plays, events or observations.
- These collections are called in this talk **teams.**
- They are the basic objects of our approach.

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- They are the basic objects of our approach.

Teams

- A set of records of stock exchange transactions of a particular dealer.
- A set of possible histories of mankind written as decisions and consequences.
- A set of chess games between Susan and Max, as lists of moves.

Teams

- 1st intuition: A team is a set of plays of a game.

Teams

- 1st intuition: A team is a set of plays of a game.
- 2nd intuition: A team is a database.

	x_0	x_1	x_2
s_0	0	1	0
s_1	0	1	1
s_2	2	5	5

Towards a logic based on teams

- A set of plays satisfies $x_2>x_0$ if move x_2 is in each play greater than move x_0.
- A set of plays satisfies $=(x_1,\ldots,x_n,y)$ if move y is in each play determined by the moves $x_1,\ldots,x_n$.

Towards a logic based on teams

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- A set of plays satisfies $=(x_1,\ldots,x_n,y)$ if move y is in each play determined by the moves $x_1,\ldots,x_n$.

- A database satisfies $x_2 > x_0$ if field x_2 is always greater than field x_0.
- A database satisfies $=(x_1,\ldots,x_n,y)$ if field y is functionally determined by the fields $x_1,\ldots,x_n$.

Dependence atoms $=(x_1,\ldots,x_n,z)$

+

First order logic

=

Dependence logic

Syntax

$=, \neg, \vee, \wedge, \exists, \forall,), (, x_i$

x_i, c, $ft_1 \ldots t_n$

$t = t'$

$=(x_1, \ldots, x_n, z)$

$Rt_1 \ldots t_n$

$t = t'$

$Rt_1 \ldots t_n$

$\neg \varphi$

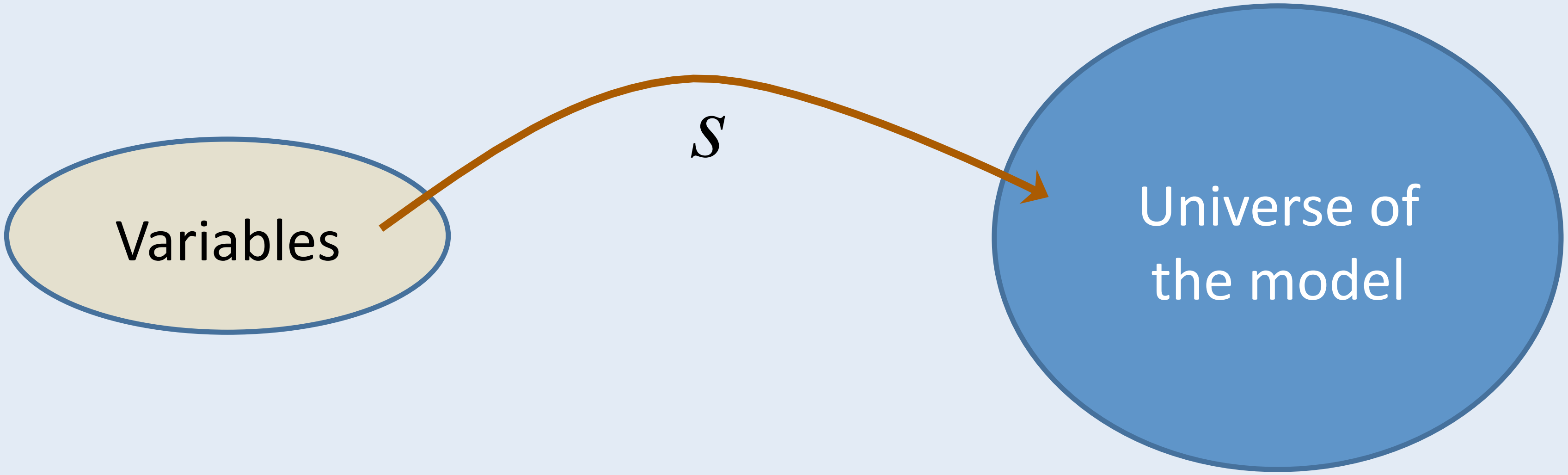
$\varphi \vee \psi$

$\varphi \wedge \psi$

$\exists x_i \varphi$

$\forall x_i \varphi$

Assignment



Teams – exact definition

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  - Database with no rows.
  - No play was played.

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- A team is just a **set** of assignments for a model.
- Empty team $\emptyset$.
  - Database with no rows.
  - No play was played.
- The team $\{\emptyset\}$ with the empty assignment.
  - Database with no columns, and hence with at most one row.
  - Zero moves of the game were played

For the truth definition: Negation Normal Form

We push negations all the way

to atomic formulas using de Morgan laws.

Thus $\neg\neg\varphi$ will have the same meaning as φ.

Truth definition

A team **satisfies a formula** if

every assignment in the team does,

and …

A team satisfies $Rt_1 \ldots t_n$ if every team member does.

	x_0	x_1	x_2
s_0	0	1	0
s_1	0	1	1
s_2	2	5	5

$x_0 < x_1$

A team satisfies $\neg Rt_1 \ldots t_n$ if every team member does.

	x_0	x_1	x_2
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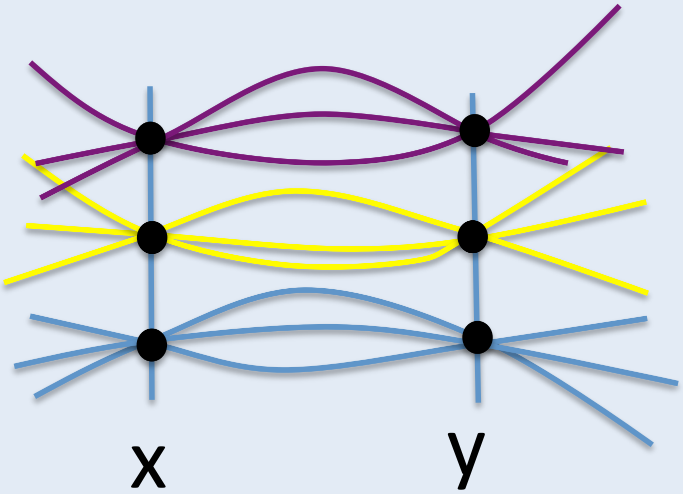
$$\neg x_1 < x_0$$

Note: some X satisfy neither $Rt_1 \ldots t_n$ nor $\neg Rt_1 \ldots t_n$.

A team satisfies t=t' if every team member does.



$x=y$

	x_0	x_1	x_2
s_0	1	0	0
s_1	0	1	1
s_2	2	5	5

$x_1=x_2$

A team satisfies ¬t=t' if every team member does.

	x_0	x_1	x_2
s_0	1	0	0
s_1	0	1	1
s_2	2	5	5

$\neg x_0 = x_1$

- A team X satisfies $=(x_1,\ldots,x_n,z)$ if in any two assignments in X, in which $x_1,\ldots,x_n$ have the same values, also z has the same value.

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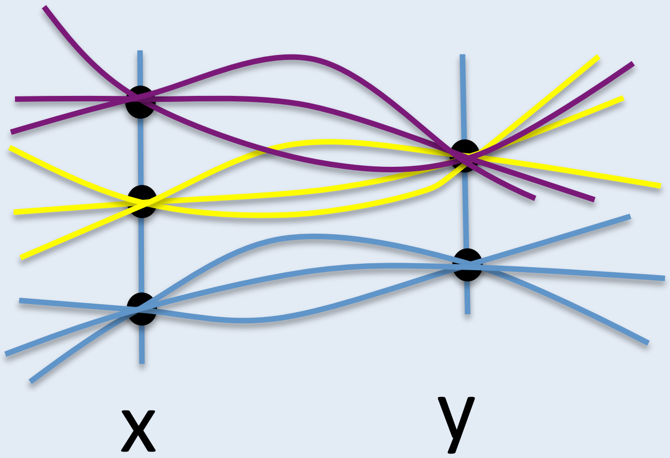

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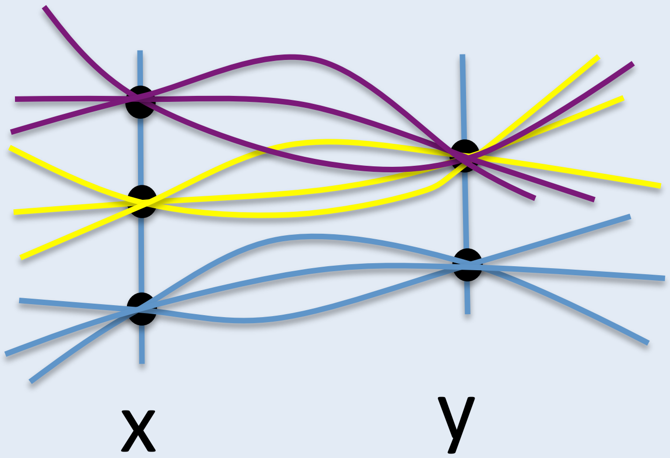

$=(x,y)$

	x	y	u	z
s_0	0	0	1	0
s_1	0	1	0	2
s_2	2	5	0	5
s_3	0	1	1	2

$=(x,y,z)$

An extreme case

=(x)

”x is constant in the team”

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=(x)

”x is constant in the team”

record	A1	A2	A3	A4	A5	A6
100000	8	6	7	3	0	6
100002	7	5	6	3	0	6
100003	4	8	7	3	0	6
100004	6	5	4	3	0	6
100005	6	12	65	3	0	6
100006	5	56	9	3	0	6
100007	6	23	0	4	0	8
...	...	...	...	...	...	...
408261	77	2	11	1	0	2

Negation of dependence atom

A team satisfies $\neg{=}(x_1,\ldots,x_n,z)$ only if it is empty.

Why?

Negation of dependence atom

A team satisfies **$\neg{=}(x_1,\ldots,x_n,z)$** only if it is empty.

Why?

Because every *singleton* team satisfies $=(x_1,\ldots,x_n,z)$, and we want downward closure (see later).

- A team X satisfies $\varphi \vee \psi$ if

  $X = Y \cup Z$, where Y satisfies φ and Z satisfies ψ.

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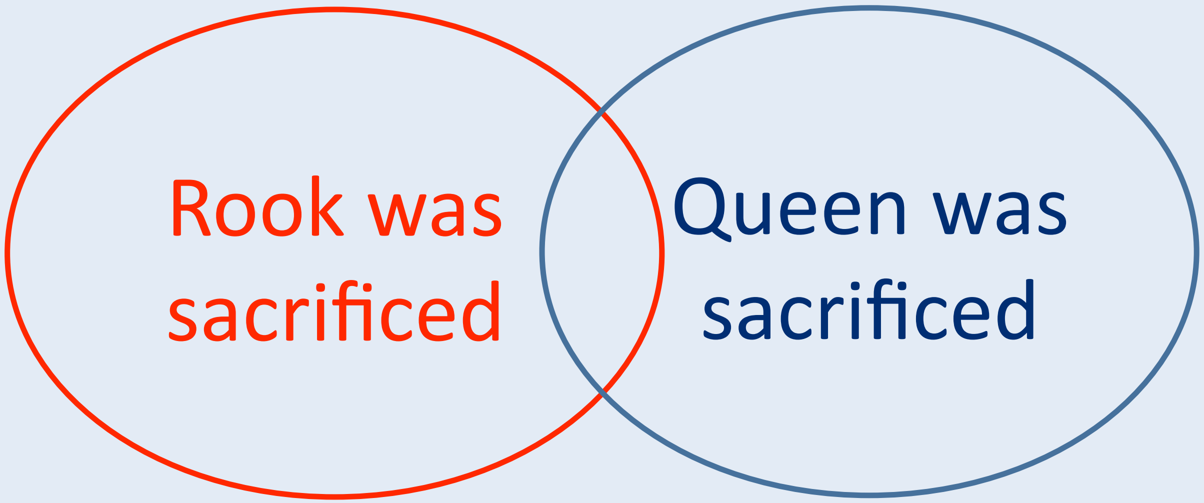
Plays where rook **or** queen was sacrificed:



Rank	Salary
A	2000
A	2100
B	2150
B	2220
C	2340
C	2440
D	2500
D	3100
E	3200
E	3710

=(Rank,Salary) ?

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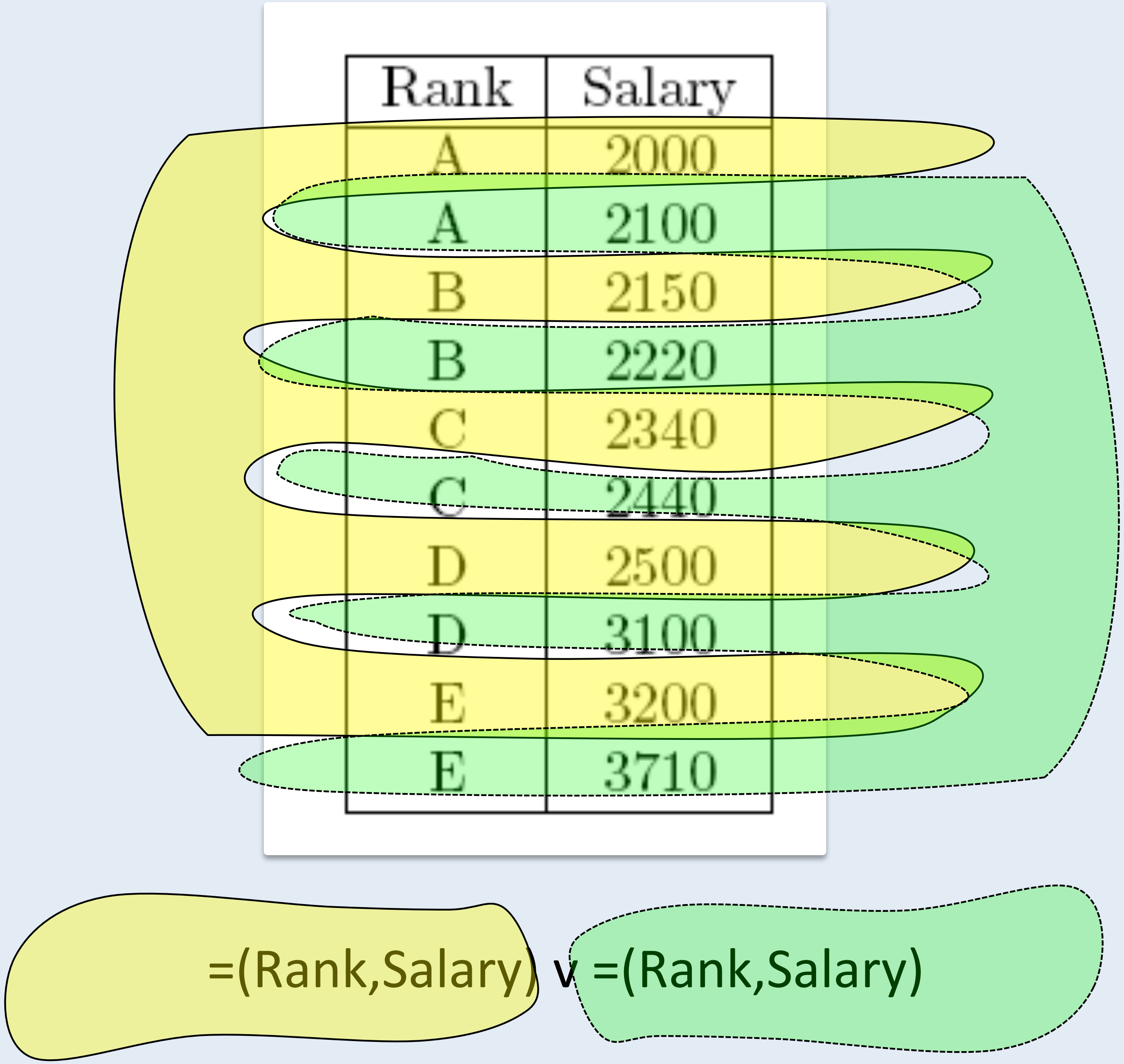
=(Rank,Salary) v =(Rank,Salary)



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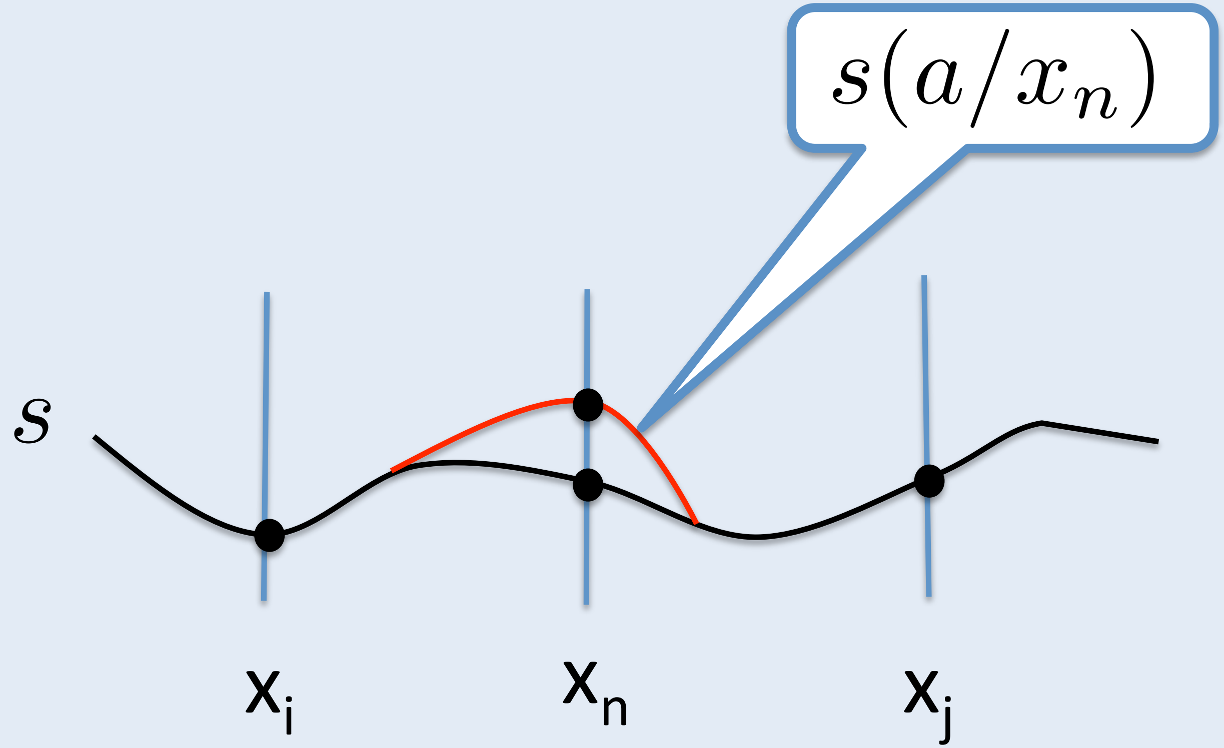
- A team X satisfies $\boldsymbol{\varphi} \wedge \boldsymbol{\psi}$ if it satisfies φ and ψ.

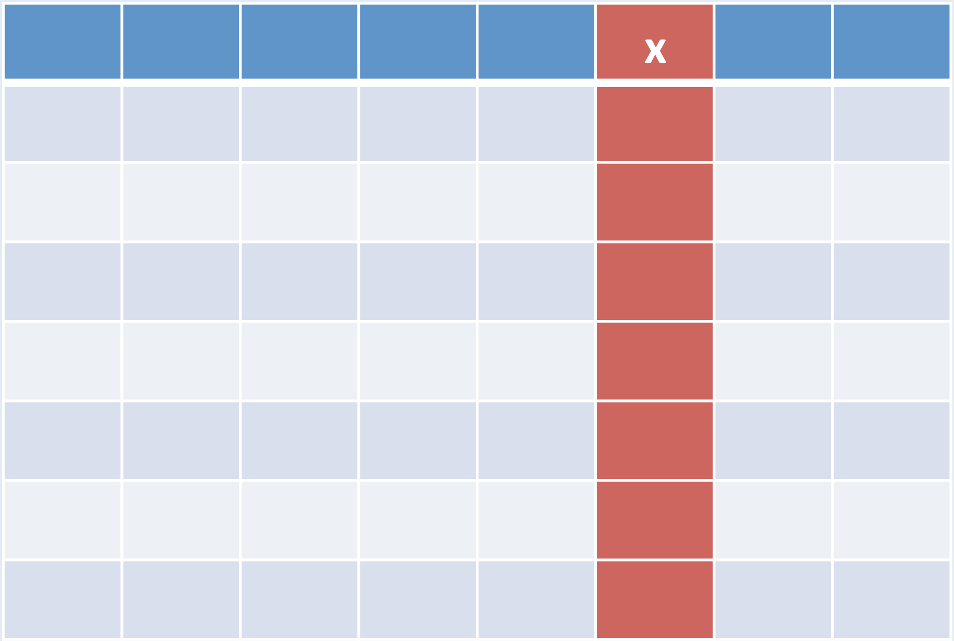
Quantifiers - modified assignment



- A team X satisfies $\exists x\varphi$ if

  there is a team Y such that Y satisfies φ and for every s in X we have $s(a/x) \in Y$ for **some** a.

Team X can be supplemented with values for x so that φ is satisfied.



X

Finnish	driver
Swedish	author
Norwegian	skier

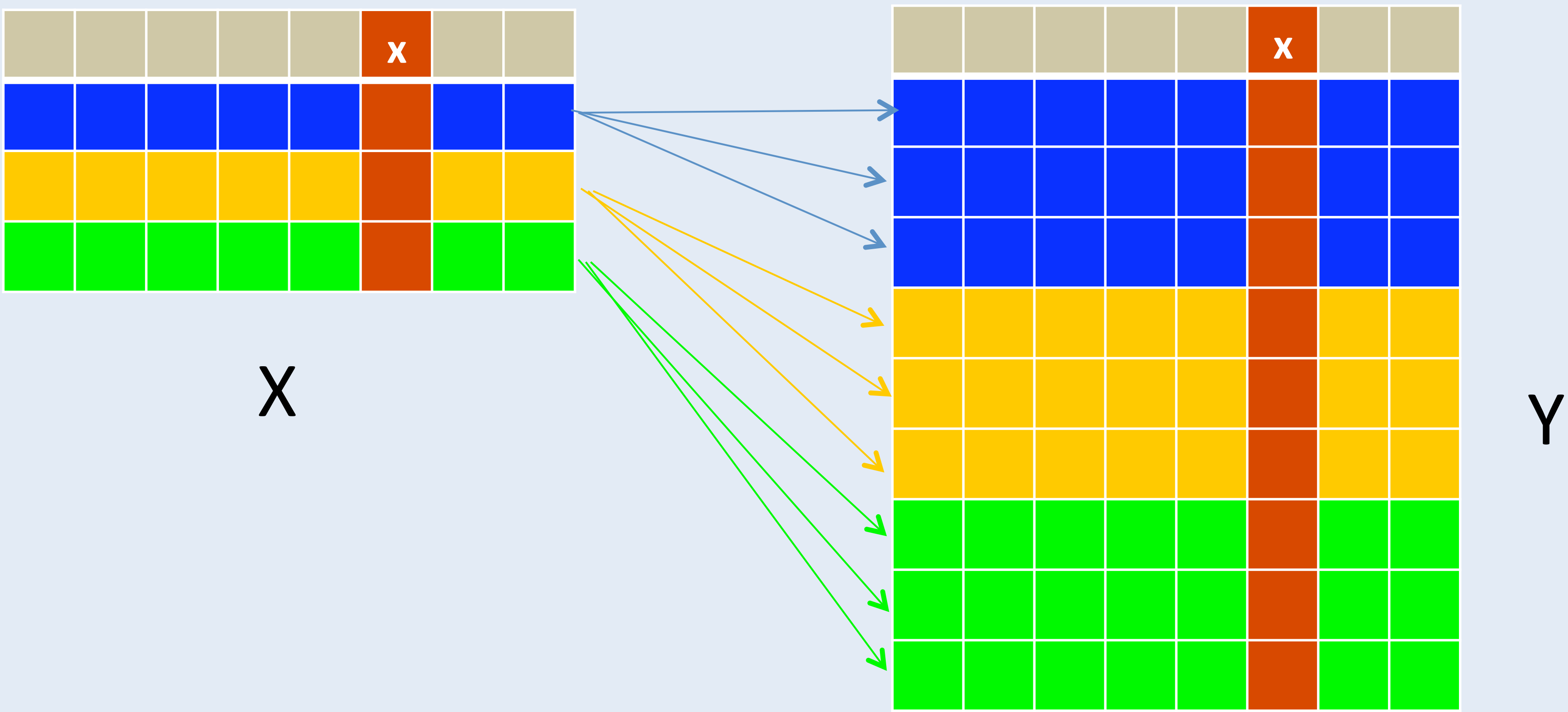
Y

		x
Finnish	driver	male
Swedish	author	female
Norwegian	skier	female

- A team X satisfies $\forall x\varphi$ if

  there is a team Y such that Y satisfies φ and for every s in X we have $s(a/x) \in Y$ for **all** a.

Team X can be duplicated along x, by letting x get all possible values, and then φ is satisfied.



X

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Y

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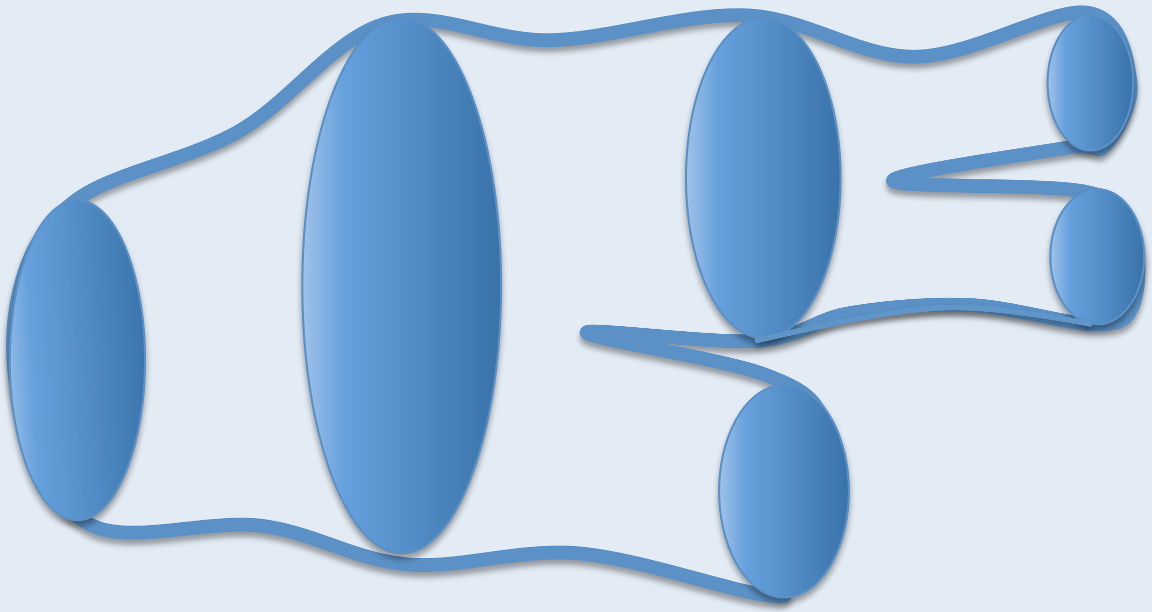
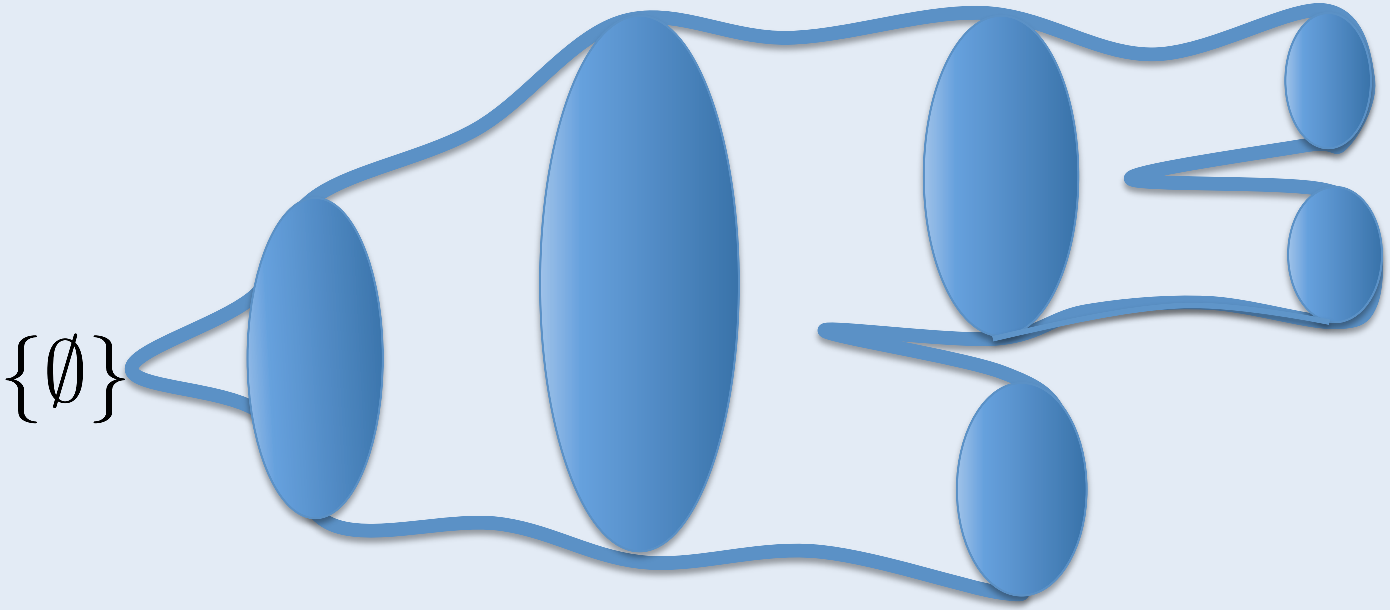


Truth

- A sentence is **true** if $\{\emptyset\}$ satisfies it.



Example: even cardinality

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 (=\!(x_2, x_3) \;\wedge\; \neg(x_0 = x_1)$$
$$\wedge\; (x_0 = x_2 \longrightarrow x_1 = x_3)$$
$$\wedge\; (x_1 = x_2 \longrightarrow x_3 = x_0))$$

Like Henkin (partially ordered) quantifiers.

Equicardinality

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 (\;\; =(x_1, y_1) \wedge$$
$$\wedge (x_0 = x_1 \leftrightarrow y_0 = y_1))$$

In inner models of set theory as strong as the unrestricted logic.

Conservative over FO

A team $\{s\}$ satisfies a **first order formula** φ

iff

s satisfies φ in the usual sense.

Two important properties

Downward closure: If a team satisfies a formula, every subset does. (Hodges: optimal!)

Empty set property: The empty team satisfies every formula.

No Law of Excluded Middle

Suppose the universe has at least two elements.

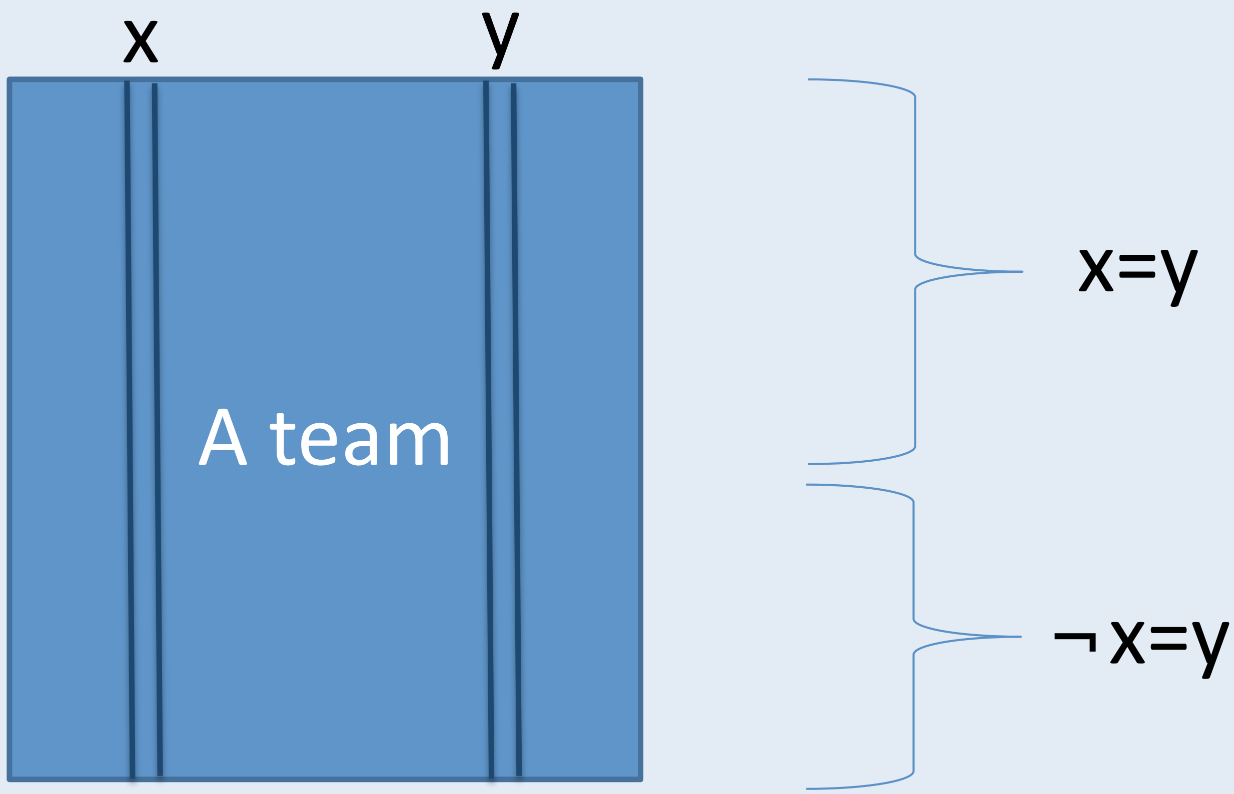
$\forall x =\!(x)$ not true

$\neg\forall x =\!(x)$ not true either

because it means $\exists x \neg{=}(x)$.

LEM holds (exactly) for the FO part

- Every team satisfies x=y v ¬ x=y:



A special axiom schema

- **Comprehension Axioms:**

$$\forall x(\varphi \vee \neg \varphi),$$

if φ is FO.

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if φ is FO.

``LEM = Comprehension Axiom”

Armstrong's Axioms

Always =(x,x)

If =(x,y,z), then =(y,x,z).

If =(x,x,y), then =(x,y).

If =(x,z), then =(x,y,z).

If =(x,y) and =(y,z), then =(x,z).

Incorrect rules

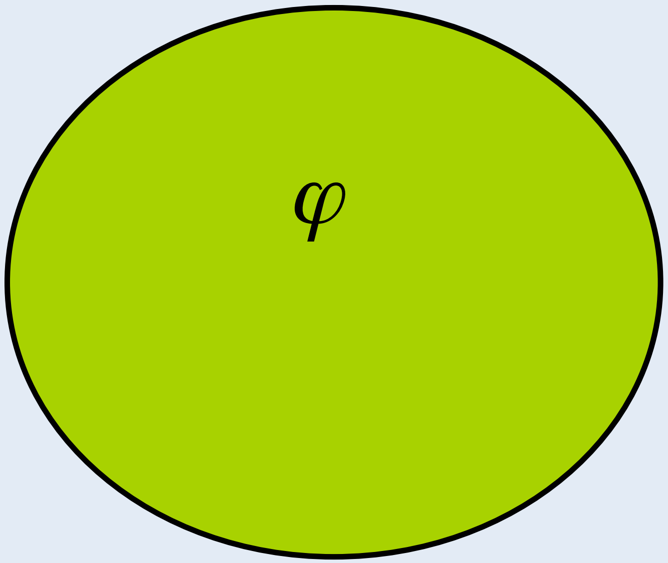
No absortion

- From $\varphi \vee \varphi$ follows φ. Wrong!
- From$(\varphi \wedge \psi) \vee (\varphi \wedge \theta)$ follows $\varphi \wedge (\psi \vee \theta)$. Wrong!
- From$(\varphi \vee \psi) \wedge (\varphi \vee \theta)$ follows $\varphi \vee (\psi \wedge \theta)$. Wrong!

Non-distributive

Example

- If $\neg\varphi \vee \psi$ is valid then φ *logically implies* ψ.



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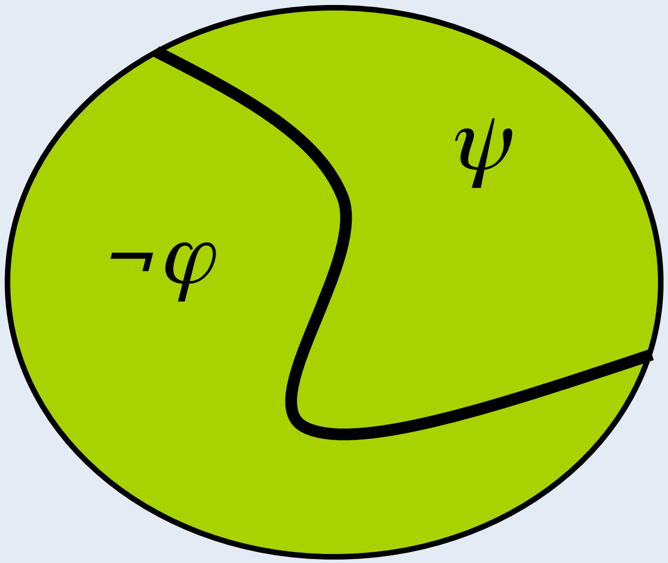

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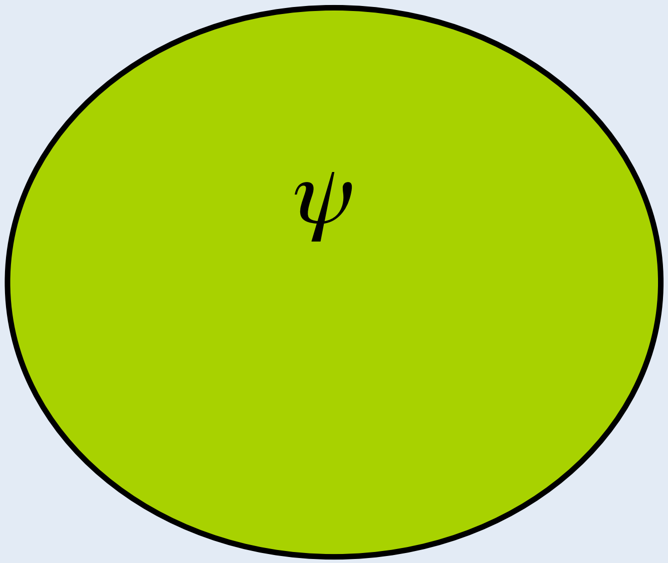

Game theoretic semantics

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  - Semantic (evaluation) game
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- Version 1: Players move assignments.
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  - Semantic (evaluation) game
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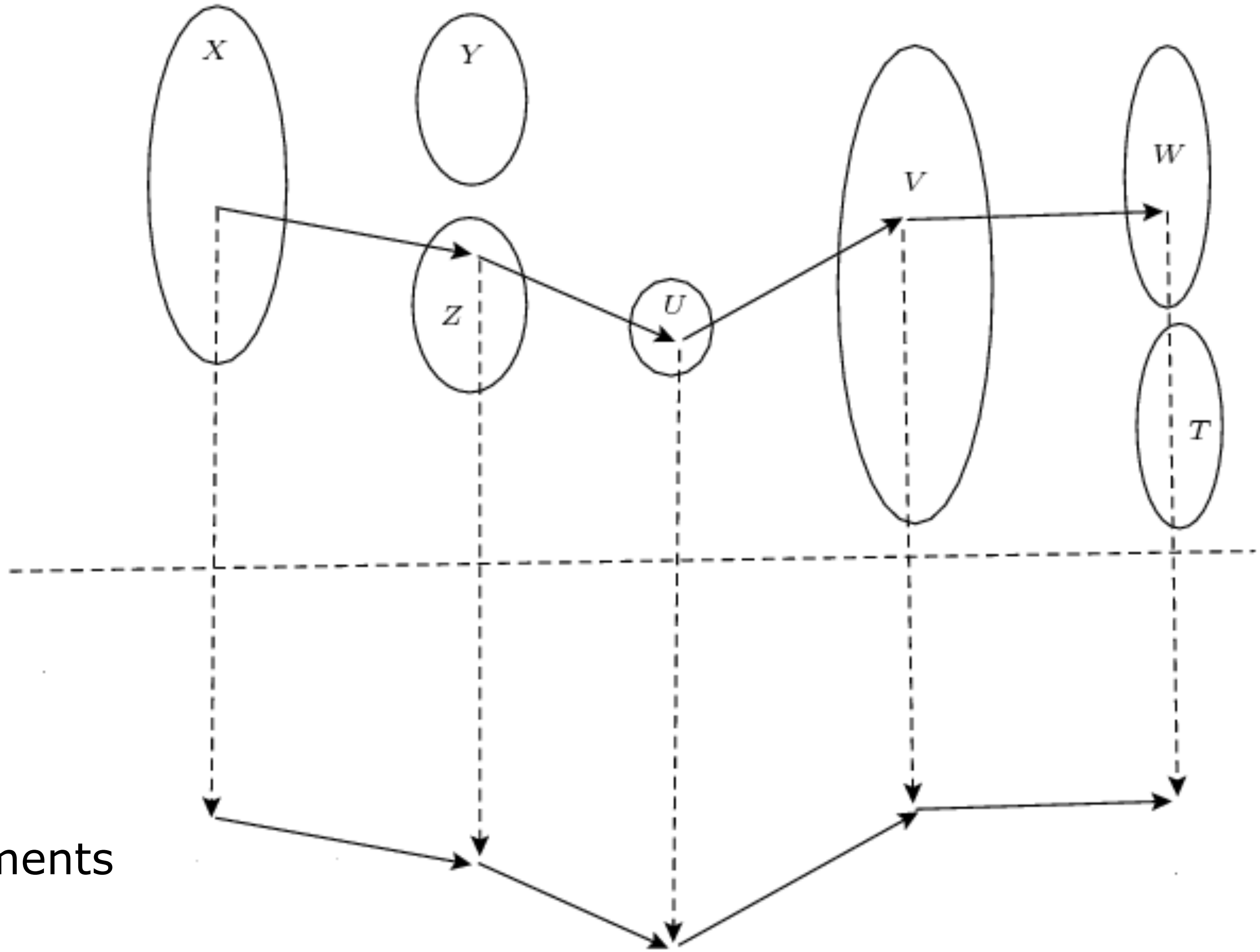
- Version 1: Players move assignments.
  - Non-deterministic, imperfect information.
- Version 2: Players move teams.
  - Deterministic, perfect information.

Teams



Assignments

Model theory of dependence logic

Hodges 1997: For every formula $\varphi(x_1,\ldots,x_n)$ there is an existential second order sentence $\Phi(P)$ with P negative such that a team X satisfies φ iff $\Phi(X)$ is true.

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Theorem: The converse is also true.

(Enderton, Walkoe 1970, Kontinen-V. 2008 - answers a question of Hodges.)

Consequences

- A language for NP on finite models.
- Compactness.
- Löwenheim-Skolem.
- Separation (Interpolation).

Hard questions

- Does φ avoid some model?

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- Σ_2-complete in set theory (V. 2001).
- Rec. isomorphic with the decision problem of second order logic.
- Π_2-concepts: GCH, SCH.

More hard questions

- Let κ be the smallest κ such that if any φ avoids some model, it avoids a model of size $<\kappa$.
- Between the first measurable and the first strong cardinal, if such exist.
- Hanf number of avoiding models even larger.
- Same with compactness of avoiding models.

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Avoid - classical negation

- The closure of dependence logic under classical negation has the exact strength of second order logic (Ville Nurmi, 2008).
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How about intuitionistic negation?

Joint work with S. Abramsky.

- *Definition*: X satisfies $\varphi \to \psi$ iff every subteam of X which satisfies φ also satisfies ψ.
- *Definition*: X satisfies $\bot$ iff X is the empty team.
- $\neg\, \varphi$ is now equivalent to $\varphi \to \bot$ for atomic φ.
- Intuitionistic negation ($\varphi \to \bot$) is an alternative way to extend negation from atomic to non-atomic formulas.

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How about intuitionistic negation?

- Dependence atoms can now be defined in terms of constancy:

$$=(x_1,\ldots,x_n,z) \equiv (=(x_1) \wedge \ldots \wedge =(x_n)) \rightarrow =(z).$$

- Downward closure and the empty set property are preserved.
- Compactness fails.
- Goes beyond NP, unless NP=co-NP.

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We can prove Armstrong's Axioms

Dependence logic	Heyting's intuitionistic logic
=(x,x)	$=(x) \rightarrow =(x)$
If =(x,y,z), then =(y,x,z).	If $(=(x) \wedge =(y)) \rightarrow =(z)$, then $(=(y) \wedge =(x)) \rightarrow =(z)$
If =(x,x,y), then =(x,y).	If $(=(x) \wedge =(x)) \rightarrow =(y)$, then $=(x) \rightarrow =(y)$
If =(x,z), then =(x,y,z).	If $=(x) \rightarrow =(z)$, then $(=(x) \wedge =(y)) \rightarrow =(z)$
If =(x,y) and =(y,z), then =(x,z).	If $=(x) \rightarrow =(y)$, and $=(y) \rightarrow =(z)$ then $=(x) \rightarrow =(z)$

Linear implication

- X satisfies φ —o ψ iff for every team Y which satisfies φ the team $X \cup Y$ satisfies ψ.
- Downward closure is preserved.
- Compactness fails.
- Goes beyond NP unless NP=co-NP.

Galois connections

- Intuitionistic implication is the adjoint of conjunction:

$$(\phi \wedge \psi) \models \theta \iff \phi \models \psi \rightarrow \theta$$

- Linear implication is the adjoint of disjunction.

$$(\phi \vee \psi) \models \theta \iff \phi \models \psi \multimap \theta$$

Proof

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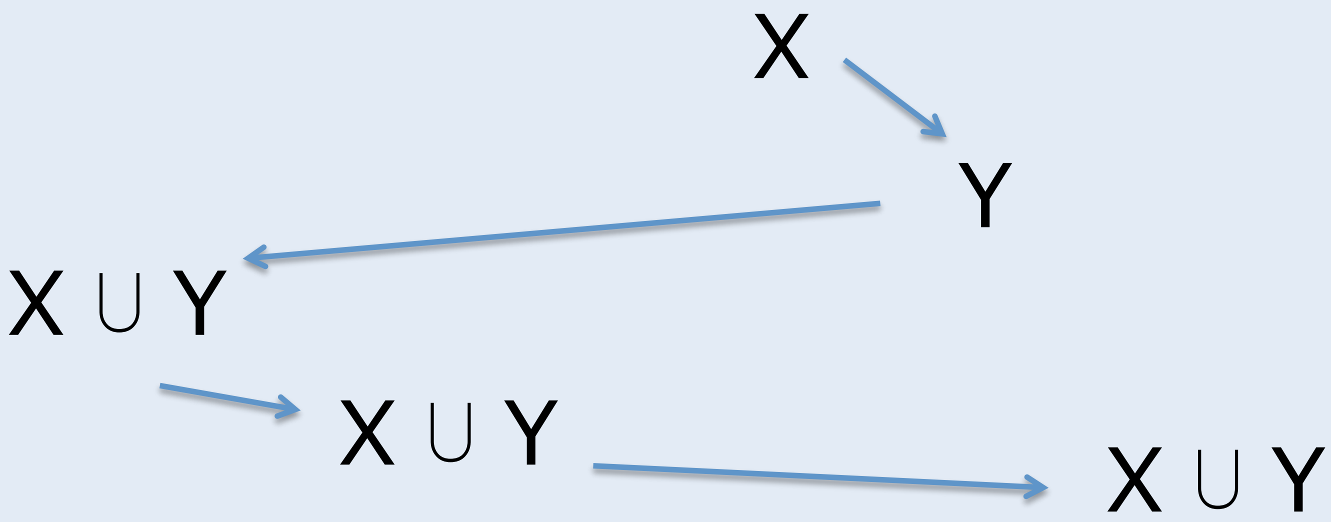
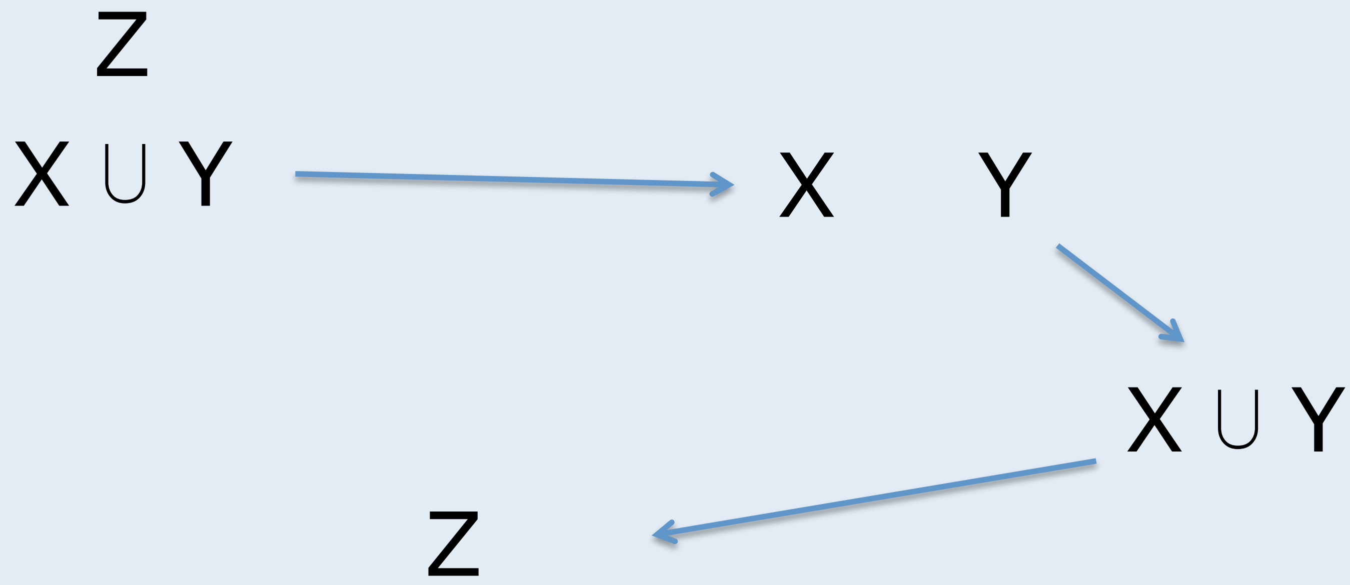
$$(\phi \vee \psi) \models \theta \iff \phi \models \psi \multimap \theta$$



Proof

- Linear implication is the adjoint of disjunction.

$$(\phi \vee \psi) \models \theta \iff \phi \models \psi \multimap \theta$$



The moral of the story

- One can add both intuitionistic and linear implication to dependence logic without losing the downward closure.
- Intuitionistic negation agrees with the original negation on the atomic level, and basic axioms of dependence become provable.
- Good (?) for proof theory, but bad (?) for model theory. Is there a reason for this?

What is dependence logic good for?

- Language for NP.
- Tool for the study of more complex dependencies than just the Armstrong ones.
- A vehicle for uncovering the mathematics of dependence in a variety of contexts
  - Data mining
  - Social choice theory
  - Quantum physics??

- Book: J. Hintikka, The Principles of Mathematics Revisited, Cambridge University Press, 1996.
- Book: J. Väänänen, *Dependence Logic*, Cambridge University Press, 2007.
- Logic for Interaction (LINT), ESF LogICCC

Thank you!