Turing Complexity of Models and Theories

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One of the directions of computable model theory is connected to the study of complexity of theories with computable models, and of algorithmic complexity of other models of such theories.
Let T be a consistent first order theory. When can T be realized in a computable model? From the other hand, how complicated is to build other models of the theory T?
In other words, what is the Turing complexity of these models?
These problems are connected to the fundamental question of existence of computable models with given model-theoretic and algorithmic properties, i.e. with given specification.

In the frame of computable model theory we study the questions of algorithmic complexity of countable models with uncountably categorical theories.
We review the results and problems of the research direction and propose solutions to some open questions.

We remind the main definitions used in the paper. Let $\mathcal{L}$ be a fixed first order language.

1. Languages are computable, and structures have universe a subset of ω.
2. We identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$.
3. We identify sentences with their Gödel numbers.

An algebraic structure (a model) $\mathcal{A}$ for this language is called **computable**, if its universe is computable and all basic operations and predicates are uniformly computable.
In other words, the atomic diagram of the model $\mathcal{A}$ is computable.
An algebraic structure $\mathcal{B}$ is **computably presentable** if it is isomorphic to a computable structure.
In this case any isomorphism of $\mathcal{B}$ onto a computable structure $\mathcal{A}$ is called a **computable presentation** of the structure $\mathcal{B}$.

An $\mathcal{L}$–structure $\mathcal{A}$ is **decidable** if its complete diagram $\mathcal{CD}(\mathcal{A})$ is computable.
We will consider relative computability and the corresponding notions of computability of structures with respect to an oracle.

Spectrum $Spec(T)$ of a theory T is
$\{a|$ any countable model of T is computable relative to degree $a\}$.

Decidable Spectrum of a theory T.

E-spectrum of T.

AutoSpectrum of a A-computable model.

A complete theory T is called **uncountably categorical** (**$\aleph_1$–categorical**) if all models of T in some uncountable power (in the power $\aleph_1$) are isomorphic.
M. Morley showed that any theory which is categorical in some uncountable power, is categorical in any uncountable power. Therefore, we will not make any difference between the two notions of uncountably categorical and $\aleph_1$–categorical theory.
A consistent theory T is called **countably categorical** (**$\aleph_0$–categorical**) if all its countable models are isomorphic.
A structure $\mathcal{M}$ is **$\aleph_1$–categorical** (**$\aleph_0$–categorical**) if its elementary theory $Th(\mathcal{M})$ is $\aleph_1$–categorical ($\aleph_0$–categorical).

In 1971 J. Baldwin and A. Lachlan showed that all countable models of an $\aleph_1$–categorical theory T, which is not countably categorical, can be listed into an infinite chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \ldots \mathcal{A}_\omega$ of elementary embeddings, where $\mathcal{A}_0$ is a prime model of the theory T, $\mathcal{A}_\omega$ is a saturated model of T and for all i the structure $\mathcal{A}_{i+1}$ is a minimal proper extension of $\mathcal{A}_i$.

The key moment in the paper of J. Baldwin and A. Lachlan was the notion of a strongly minimal formula and the possibility to build an operator *acl* of closure on the set definable by this formula. This operator is analogous to the operator of algebraic closure in fields.

A formula $\varphi(x)$ is called minimal in a structure $\mathcal{M}$ of T if the set $\varphi(\mathcal{M}) \rightleftharpoons \{a \in |\mathcal{M}| \, | \mathcal{M} \vDash \varphi(a)\}$ is infinite but can not be devided into two infinite parts by any formula for the language of T with parameters in $\mathcal{M}$.

A formula $\varphi(x)$ is called strongly minimal in T if $\varphi(x)$ is minimal in all models $\mathcal{M}$ of T.

For a strongly minimal formula $\varphi(x)$ on subset X of the set $\varphi(\mathcal{M})$, for every structure $\mathcal{M}$ of this theory, the value *acl*(X) of the operator *acl* equals the union of all finite definable subsets of $\varphi(\mathcal{M})$ with parameters from X.
This operator acts similarly to the operator of the algebraic closure in fields.
We can define the notion of a basis relative to this operator.
The cardinality of the basis defines the isomorphism type of structures of uncountably categorical theories up to isomorphism.

One of the important subclasses of uncountably categorical theories is the class of strongly minimal theories.
A complete theory T is *strongly minimal* if for every its model, every definable with parameters in this model subset of the domain of this model is finite or cofinite.
In other words, the identically true formula is strongly minimal in this theory.
A strongly minimal theory is called trivial (more precisely, has trivial geometry) if for every subset $A \subseteq M$, the following equality holds:

$$\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(\{a\}).$$

It is well-known theorem of Yu.Ershov that a decidable theory T has a decidable model for which the satisfactory predicate is decidable.
On the other hand, if a theory T has a computable model then T is decidable relative to $\mathbf{0}^{\omega}$.
For example, the theory of arithmetic $(\omega, S, +\times, \leq, 0)$ is Turing equivalent to the set $\mathbf{0}^{\omega}$ but its standart model is computable. However, all its other countable models have nonarithmetical complexity.
In addition, there are examples of Tannenbaum of finitely axiomatizable, hence, computably enumerable theories without computable models.

Categorical in some power theories are a classical object of model theory.
We define a spectrum of computable models $SCM(T)$ of the theory T. We let
$SCM(T) = \{i \mid \mathcal{A}_i$ has a computable presentation$\}$.
If a theory T is $\aleph_1$–categorical and decidable, then L. Harrington and N.G. Khisamiev in 1974 showed that all countable models of T have computable presentations.
Thus, in this case $SCM(T) = \omega \bigcup \{\omega\}$.

In 1978 by me was proved the existence of an $\aleph_1$–categorical theory T which is $\mathbf{0}'$–computable and $SCM(T) = \{0\}$.

K. Kudaibergenov in 1980 generalized this result and showed that for every $n \geq 0$ there exists an $\aleph_1$–categorical $\mathbf{0}'$–computable theory T, such that $SCM(T) = \{0, 1 \ldots, n\}$.

A.Nies showed that there exists a theory with $SCM(T_1) = \{1\}$.
B. Khoussainov, A. Nies and R. Shore showed the existence of two $\aleph_1$–categorical $\mathbf{0}''$–computable theories T_1 and T_2, such that $SCM(T_1) = \omega$ and $SCM(T_2) = \omega \bigcup \{\omega\} \setminus \{0\}$.

In 2002 S. Goncharov and B. Khoussainov showed that for every natural $n \geq 1$, there exists an $\aleph_1$–categorical theory T in a finite language with a computable model, such that T is Turing equivalent to $\mathbf{0}^n$.
Moreover, all countable models of T has computable presentations and T is almost strongly minimal.

In 2005 E. Fokina generalized this result and showed that for every arithmetical Turing degree a there exists an $\aleph_1$–categorical theory T in a finite language with a computable model, such that T is Turing equivalent to a.

We note that the following results for strongly minimal theories with trivial pregeometry were received by Goncharov S., Harizanov V., Lempp S., Laskowski M., McCoy Ch. F. D. in 2003.

The research of the Turing complexity of countable models of strongly minimal theories was initiated by S. Lempp and was conducted in the frame of the grant of NSF USA number DMS-0075899.

Theorem

2003, by Goncharov S., Harizanov V., Lempp S., Laskowski M., McCoy Ch. F. D.
The theory $\mathrm{Th}(\mathcal{M})$ *of a computable strongly minimal structure* $\mathcal{M}$ *with trivial geometry is* $\mathbf{0}''$*–decidable. Moreover, all countable models of* $\mathrm{Th}(\mathcal{M})$ *are* $\mathbf{0}''$*–decidable, in particular,* $\mathbf{0}''$*–computable.*

As mentioned above, an important role in the proof of the properties play the method of the pure model theory.

Theorem

2003, by Goncharov S., Harizanov V., Lempp S., Laskowski M., McCoy Ch. F. D.
For every strongly minimal $\mathcal{L}$–theory with trivial geometry, the elementary diagram of any its model $\mathcal{M}$ is a model complete $\mathcal{L}_M$–theory. Any strongly minimal $\mathcal{L}$–theory with trivial geometry is $\exists\forall\exists$–axiomatizable.

These results became the basis for new hypotheses about the complexity of the construction of strongly minimal theories in general case. In connection with these problems it was possible to show that the following result is true for strongly minimal theories in general case.

Theorem

If $\mathcal{M}$ is an a–computable model of a strongly minimal theory, then all its countable models are a''–computable.

Algorithm to verify independents $Bound_m^n$

Let $\mathfrak{M}$ be a–computable model of strongly minimal theory T. Let $a_1, ..., a_n$ be a sequence of different elements from some basis of countable saturated model $\mathfrak{M}_\omega$ of our theory T. We define algorithm with oracle $a^{(m+2)}$ to verify in strongly minimal theory T that a formula $\varphi(x_1, ..., x_n)$ with n free variables $x_1, ..., x_n$ and in prenex normal form with $(m+1)$-bloc of homogeneous quantifiers and started with $\exists$ will be holds on $a_1, ..., a_n$ in this countable saturated model $\mathfrak{M}_\omega$.

Algorithm $Bound_m^n$

Step 1. For formula $\varphi(x_1, ..., x_n)$ with n free variables $x_1, ..., x_n$ and in prenex normal form with $(m+1)$-bloc of homogeneous quantifiers and started with $\exists$ we consider a formula $\varphi'(x_1, ..., x_n, y_1, ..., y_s)$ in prenex normal form with (m)-bloc of homogeneous quantifiers such that $\varphi(x_1, ..., x_n)$ equivalent a formula $(\exists y_1)...(\exists y_s)\varphi'(x_1, ..., x_n, y_1, ..., y_s)$.

Step 2. We will compute with oracle $a^{(m+1)}$ the list $(l_1, ..., l_n)$ of natural numbers from substeps $0 \leq i \leq n$:
Substep n. We can verify for any $\ell \in \omega$ with oracle $a^{(m+1)}$ the truth in a–computable model $\mathfrak{M}$ of strongly minimal theory T of formula

$$(\exists y_1)...(\exists y_s)(\exists x_1, \ldots, \exists x_{n-1})((\exists^{\geq \ell} x_n)\varphi'(x_1, ..., x_n) \,\&$$

$$(\exists^{\geq \ell} x_n)\neg\varphi'(x_1, ..., x_n))$$

.
From strongly minimality of the theory T we can find minimal $\ell_n \in \omega$ such that correspondent formula is not holds.

Now we can check the truth with oracle $a^{(m+1)}$ the truth in a–computable model $\mathfrak{M}$ of strongly minimal theory T of formula

$$(\exists y_1)...(\exists y_s)(\exists x_1, \ldots, \exists x_{n-1})((\exists^{\geq \ell_n} x_n)\varphi'(x_1, ..., x_n)$$

. If this formula fails then our formula $\varphi(x_1, ..., x_n)$ will fails on elements $a_1, ..., a_n$.

If this formula holds then we will go to the net substep.

Substep $n-1$. Simillary we can compute with oracle $a^{(m+1)}$ a minimal ℓ_{n-1} such that the formula

$$(\exists y_1)...(\exists y_s)(\exists x_1,\ldots,x_{n-2})((\exists^{\geq \ell_{n-1}} x_{n-1})(\exists^{\geq \ell_n} x_n \varphi')$$

$$\vee(\exists^{\geq \ell_{n-1}} x_{n-1})\neg(\exists^{\geq \ell_n} x_n)\varphi'))$$

is not holds.
We can note easy that the formula

$$(\exists y_1)...(\exists y_s)(\exists x_1,\ldots,x_{n-2})((\exists^{\geq \ell} x_{n-1})(\exists^{\geq \ell_n} x_n \varphi')\vee(\exists^{\geq \ell} x_{n-1})\neg(\exists^{\geq \ell_n} x_n)\varphi$$

holds iff
the formula

$$(\exists y_1)...(\exists y_s)(\exists x_1,\ldots,x_{n-2})((\exists^{\geq \ell} x_{n-1})(\exists^{\geq \ell_n} x_n)\varphi')$$

$$\vee(\exists^{\geq \ell} x_{n-1})(\exists^{\geq \ell_n} x_n)\neg\varphi')),$$

holds.

We have this equivalence from equivalence of formulas $\neg\exists^{\geq \ell_n} x_n \varphi$ and $\exists^{\geq \ell_n} x_n \neg\varphi_n$ from the properties of elelement ℓ_n.

Now we can check the truth with oracle $a^{(m+1)}$ the truth in a–computable model $\mathfrak{M}$ of strongly minimal theory T of formula

$$(\exists y_1)...(\exists y_s)(\exists x_1, \dots, \exists x_{n-2})(\exists^{\geq \ell_{n-1}} x_{n-1})((\exists^{\geq \ell_n} x_n)\varphi'(x_1, ..., x_n)$$

. If this formula fails then our formula $\varphi(x_1, ..., x_n)$ will fails on elements $a_1, ..., a_n$.

Substep i. Let $0 \le i < n$ and from previous steps we found elements $\ell_{n-i+1}, \dots, \ell_n$ such that for any $n > j \ge i$ with oracle $a^{(m+1)}$ the truth in a–computable model $\mathfrak{M}$ of strongly minimal theory T of formulas

$$\neg(\exists y_1)...(\exists y_s)(\exists x_1, \dots, \exists x_1 x_j)$$

$$((\exists^{\ge \ell_{j+1}} x_{j+1})(\exists^{\ge \ell_{j+2}} x_{j+2}) \dots (\exists^{\ge \ell_n} x_n)\varphi' \vee (\exists^{\ge \ell_{j+1}} x_{j+1})$$

$$\neg(\exists^{\ge \ell_{j+2}} x_{j+2})(\exists^{\ge \ell_{i+3}} x_{i+3}) \dots (\exists^{\ge \ell_n} x_n)\varphi'),$$

and each time we compute minimal element with this property.

We will check for any natural number ℓ and find the first ℓ_i such that the formula

$$\neg(\exists y_1)...(\exists y_s)$$
$$(\exists x_1, \ldots, x_{i-1})((\exists^{\geq \ell} x_i)(\exists^{\geq \ell_{i+1}} x_{i+1})(\exists^{\geq \ell_{i+2}} x_{i+2}) \ldots (\exists^{\geq \ell_n} x_n)\varphi'$$
$$\vee(\exists^{\geq \ell} x_i)\neg(\exists^{\geq \ell_{i+1}} x_{i+1})(\exists^{\geq \ell_{i+2}} x_{i+2})(\exists^{\geq \ell_{i+3}} x_{i+3}) \ldots (\exists^{\geq \ell_n} x_n)\varphi')$$

holds in a–computable model $\mathfrak{M}$ of strongly minimal theory T.

From strongly minimality and properties of $\ell_{n-i+1}, \ldots, \ell_n$ we have by induction the equivalence with formula

$$\neg(\exists y_1)...(\exists y_s)$$

$$(\exists x_1, \ldots, x_{i-1})((\exists^{\geq \ell} x_i)(\exists^{\geq \ell_{i+1}} x_{i+1})(\exists^{\geq \ell_{i+2}} x_{i+2}) \ldots (\exists^{\geq \ell_n} x_n)\varphi'$$

$$\vee(\exists^{\geq \ell} x_i)(\exists^{\geq \ell_{i+1}} x_{i+1})(\exists^{\geq \ell_{i+2}} x_{i+2})(\exists^{\geq \ell_{i+3}} x_{i+3}), \ldots (\exists^{\geq \ell_n} x_n)\neg\varphi').$$

We can verify the truth with oracle $a^{(m+1)}$.

It give us possibility to find on substep *i* a minimal number ℓ_i such that in our *a*-computable model the formula

$$\neg(\exists x_1, \ldots, x_{i-1})$$
$$((\exists^{\geq \ell_i} x_i)(\exists^{\geq \ell_{i+1}} x_{i+1}) \ldots (\exists^{\geq \ell_{i+2}} x_{i+2})(\exists^{\geq \ell_n} x_n)\varphi'$$
$$\vee(\exists^{\geq \ell_i} x_i)\neg(\exists^{\geq \ell_{i+1}} x_{i+1})(\exists^{\geq \ell_{i+2}} x_{i+2})(\exists^{\geq \ell_{i+3}} x_{i+3}) \ldots (\exists^{\geq \ell_n} x_n)\varphi')$$

holds.

On the base of this construction we compute with oracle $a^{(m+1)}$ a sequence of natural numbers $\ell_1, \ell_2, \dots, \ell_n$ such that for any $0 \leq i < n$ the formula

$$\neg(\exists x_1, \dots, x_i)((\exists^{\geq \ell_{i+1}} x_{i+1})(\exists^{\geq \ell_{i+2}} x_{i+2}) \dots (\exists^{\geq \ell_n} x_n)\varphi'$$

$$\vee(\exists^{\geq \ell_{i+1}} x_{i+1})\neg(\exists^{\geq \ell_{i+2}} x_{i+2})(\exists^{\geq \ell_{i+3}} x_{i+3}) \dots (\exists^{\geq \ell_n} x_n)\varphi')$$

holds.

For this sequence the formula

$$(\exists^{\geq \ell_1} x_1)(\exists^{\geq \ell_2} x_2) \dots (\exists^{\geq \ell_n} x_n)\varphi$$

holds in $\mathfrak{M}$ iff the formula $\varphi(x_1, ..., x_n)$ holds on the basic elelments $a_1, ..., a_n$ in saturated model $\mathfrak{M}_\omega$.

Lemma

There exists an algorithm with oracle a^{m+1} *for any formula* $(\exists y_1)...(\exists y_1)\varphi(x_1, x_2, ..., x_n, y_1, ..., y_k)$ *where* $\varphi(x_1, x_2, ..., x_n, y_1, ..., y_k) \in \Pi_m$ *that the formula*

$$(\exists y_1)...(\exists y_1)(Q_1 x_1)(Q_2 x_2)\dots(Q_n x_n)\varphi$$

holds in $\mathfrak{M}$ *if* $\varphi(x_1, x_2, ..., x_n, y_1, ..., y_k)$ *is* Π_m*-formula in prenex normal form where* $Q_i \in \{\exists^\infty, \exists\}$*.*

Lemma

There exists an algorithm with oracle a^2 for any boolean combination $\varphi(x_1, x_2, ..., x_n, y_1, ..., y_k)$ of existential formulas that the formula

$$(Q_1 x_1)(Q_2 x_2) \ldots (Q_n x_n)\varphi$$

holds in $\mathfrak{M}$ where $Q_i \in \{\exists^\infty, \exists\}$.

Algorithm $Boolean - Bound_m^n$

There exists an algorithm with oracle a^{m+2} for any formula $(\exists y_1)...(\exists y_1)\varphi(x_1, x_2, ..., x_n, y_1, ..., y_k)$ where $\varphi(x_1, x_2, ..., x_n, y_1, ..., y_k) \in \Pi_m$ and boolean combination $\gamma(x_1, x_2, ..., x_n, y_1, ..., y_k)$ of existential formulas can compute that there exist a boolean combination $\psi(x_1, x_2, ..., x_n)$ of existential formulas and ℓ such that the formulas

$$(\exists y_1)...(\exists y_k)(Q_1 x_1)(Q_2 x_2)\dots(Q_n x_n)\varphi$$

,

$$(\exists y_1)...(\exists y_1)(\exists x_1)(\exists x_2)\dots(\exists x_n)(\varphi \& \psi)$$

and

$$(\forall y_1)...(\forall y_k)(\forall x_1)(\forall x_2)\dots(\forall x_n)(psi \to \gamma)$$

hold in $\mathfrak{M}$ where $\varphi(x_1, x_2, ..., x_n, y_1, ..., y_k)$ is Π_{m+1}-formula in prenex normal form and where $Q_i \in \{\exists^\infty, \exists\}$.

We consider three cases for this proof.
Case 1. Let $\mathfrak{M}_n$ be a model with basis from n elements and there are a basis $a_1, ..., a_n$ and a finite subset A such that for any element b from $M_n \setminus A$ there exist boolean combination $\gamma(x_1, x_2, ..., x_n, \overline{z}, y)$ of existential formulas such that we have only finite set of elements y in $\mathfrak{M}$ such that
$\mathfrak{M} \models \gamma(a_1, x_2, ..., a_n, A, y)$ and $\mathfrak{M} \models \gamma(a_1, x_2, ..., a_n, A, b)$.

Case 2. Let $\mathfrak{M}_n$ be a model with basis from n elements and there are basis $a_1, ..., a_n$ and there is not a finite subset A such that for any element b from $M_n \setminus A$ there exist boolean combination $\gamma(x_1, x_2, ..., x_n, y)$ of existential formulas such that we have only finite set of elements y in $\mathfrak{M}$ such that $\mathfrak{M} \models \gamma(a_1, x_2, ..., a_n, A, y)$ and $\mathfrak{M} \models \gamma(a_1, x_2, ..., a_n, A, b)$.
Case 3. Let $\mathfrak{M}_\omega$ be saturated model.

To prove the theorem we use a relativized version of the K. Ash's theorem. It allows us to build enumerable sets satisfying a hyperarithmetical specification.

Here is a relativized version of the Ash's theorem.

Theorem (Ash- Knight)

If $\mathfrak{B} = (L, U, \hat{\ell}, P, E, (\leq_\beta)_{\beta<\alpha})$ is an α–X–system, then for every Δ^0_α–instruction q there exists a path in (P, q), such that $E(\pi)$ is computably enumerable with the oracle X. Moreover, using the $\Delta^{0,X}_\alpha$–index for q and all computably enumerable relative to X indices of the components of α–X–system $\mathfrak{B}$, we can compute a Δ^0_α index for π and a computably enumerable with the oracle X index for $E(\pi)$.

We apply this theorem with $\alpha = \omega$ and a set in the degree a'' as an oracle.

Case 1. Let A be a set with minimal the number of elements such that we have case 1. Let $A = \{d_1, ..., d_l\}$ and $\Phi(x_1, ..., x_n, z_1, ..., z_l)$ be a formula such that $\Phi(a_1, ..., a_n, z_1, ..., z_l)$ is a complete formula and $\mathfrak{M}_n \models \Phi(a_1, x_2, ..., a_n, d_1, ..., d_l)$.

Let be σ_ω consist all symbols of signature of model $\mathfrak{M}$ and new constants $a_1, ..., a_n$, $d_1, ..., d_l$ and $c_0, ..., c_n, ...$.

Let $\{\gamma_0, ..., \gamma_n, ...\}$ be all boolean combinations of existential sentences of signature σ_ω.
Let $\{\Phi_0, ..., Phi_n, ...\}$ be all sentences of signature σ_ω in prenex normal form.

Let P be the set of finite sets p finite boolean combinations of existential sentences of signature σ_ω such that

$$(\exists^{<\infty} y_1)...(\exists^{<\infty} y_k)(\exists^{\infty} x_1)(\exists^{\infty} x_2)\ldots(\exists^{\infty} x_n)(\exists^{\infty} z_1)(\exists^{\infty} z_2)\ldots(\exists^{\infty} z_l)\varphi_p$$

hold in $\mathfrak{M}$ where φ_p is $[\& p]^{\varphi_a,\varphi_d,\varphi_c}_{\varphi_x,\varphi_z,\varphi_y}$.

We define an ω-system. Let U be a finite sequence $T_0, ... T_s$ of finite sequences $T_i = (\Psi, \varphi, \vee_r [\Psi']^y_{c_{j_r}}, \varphi', S, c_{j_1}, ..., c_{j_t}, u_i)$ if Ψ is formula form $\Psi = (\exists y)\Psi'$ or Ψ, φ, S, u_i in opposite case, u_i finite sets of sentences of signature σ where u_i consist formulas defined connections with previous form like in $Bound^n_m$ and in $Boolean - Bound^n_m$ but $\varphi \in P$ and $\varphi' \in P$.

Let L consist of finite sequences
$l = p_0 u_0 p_1 u_1 ... u_r p_r$
satisfying the following conditions:
(1)$p_0 = \emptyset$,
(2) $u_0 = \{\Phi(a_1, x_2, ..., a_n, d_1, ..., d_l)\}$,
(3) $u_i \subseteq u_{i+1}$ and $u_i \subseteq u_{i+1}$,
(4) $(\cup u_i) \cup (\cup p_i)$ is consistent,
(5)for any $i < r$ the formula equivalent to Φ_i or $\neg\Phi_i$ in prenex form is first element in T_i^r.
Let $E(l) = \{\theta | p_r \vdash \theta\}$.

Let $l = p_0 u_0 p_1 u_1 ... u_r p_r$ $l' = p'_0 u'_0 p'_1 u'_1 ... u'_{r'} p'_{r'}$.
Let $l \leq_i l'$ if $\exists \overline{y'} \& p'_{r'} \rightarrow p_r$

Now we define a tree D as a set of finite sequences
$\rho = \widehat{l_0 u_0} u_0 l_1 u_1 ...$ such that
(1) $u_k \in U$,
(2) $l_k \in L(u_k)$,
(3) $u_k \sqsubseteq u_{k+1}$,
(4)$l_k \leq_0 l_{k+1}$,
(5) if $l_k = p_0 w_0 p_1 w_1 ... w_r p_r$, then either
(5.1)$l_k \in L(u_{k+1})$ and $l_{k+1} = p_0 w_0 p_1 w_1 ... w_{r-1} p_r w_r p_{r+1}$ or else
(5.2)$l_k \notin L(u_{k+1})$ and $l_{k+1} = p_0 w_0 p_1 w_1 ... w_n p^*_{r+1}$, where condition from (u_n) is different on level n from w_n.
(5.3)If we have conditions on $\Phi_r = (\exists y)\Phi'_r$ with boolean combination $(\exists y)\varrho_i$ and $(\exists y)\varrho_r$ is consistent with p_{r+1} then there exists $c_j$such that $[\varrho_i]^y_{c_j} \in p^*_{r+1}$.

Let $L(u)$ be a set l from L such that all conditions from T_i are consistent with u.
We define instruction q on the base algorithms $Bound_m^n$ and $Boolean - Bound_m^n$.

Case 2 is similar to case 1 if the answer on $Boolean - Bound^n_m$ negative we mark the corespondent new constant by $+$ and we will change definition of $\leq_i$ such that for any constant with $+$ in I we have $+$ in another I' on this constant too. For constants with $+$ in p we will have requirement about infinity of set of elements for corespondent formula.

Case 3 we can do in the same way.

Corollary

If $\mathfrak{M}$ computable model of strongly minimal theory then all countable models of this theory are 0^2–computable.

Theorem (Khoussainov B., Lempp S., Laskowski M., Solomon R.)

There exists a strongly minimal theory with computable prime model but all non prime model computable exactly relative to oracle $0^{(2)}$.

Theorem

If $\mathfrak{M}$ computable model of uncountable categorical theory then there exists a k such that all countable models of this theory are 0^k–computable.

The complexity of this models dependent from complexity of strongly minimal formula and Morley Rank of this theory, Thanks a lot for attention.