Strongly-Bounded-Turing Degrees of C.E. Sets (Work in Progress)

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CONTENTS

Bounded Turing reducibilities

Notions of bounded Turing reducibilites have been introduced by imposing bounds $b(x)$ on the admissible sizes of the oracle queries in a Turing reduction $A(x) = \Phi^B(x)$:

- $b(x)$ computable: bounded Turing (bT) or weak truth table (wtt)
- $b(x) = id(x) = x$: identity bounded Turing (ibT)
- $b(x) = id(x) + c = x + c$: strongly bounded Turing (sbT) or strong weak truth table (sw) or computable Lipschitz (cl)

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For $b(x) = id(x) + k$ we call the corresponding reducibility also $(i + k)bT$ (which, for $k \geq 1$, is not transitive).

Examples of *ibT* reductions

Some typical, frequently used examples of *ibT* reductions on the c.e. sets are the following.

- (Delayed) Permitting

$$x \in A_{at\ s} \;\Rightarrow\; \exists\, y \leq x\, (y \in B_{at\ s})$$

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- Splitting

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- Splitting

$$A = B \,\dot\cup\, C \ \Rightarrow\ B \le_{ibT} A \ \text{ and } \ C \le_{ibT} A$$

In fact,

$$deg_{ibT}(A) = deg_{ibT}(B) \vee deg_{ibT}(C)$$

Origins and Applications

- *ibT*-reducibility was introduced by Soare (2004) in the context of some applications of computability theory to differential geometry (Nabutovski and Weinberger).
- *sbT*-reducibility was introduced by Downey, Hirschfeldt and LaForte (2001) in the context of computable randomness.

The structure of these reducibilities on the c.e. sets (and c.e. reals) has been recently studied by Barmpalias, Belanger, Csima, Ding, Downey, Fan, Hirschfeldt, LaForte, Lewis, Soare, Yu, and others.

Strongly bounded Turing reducibility and computable invariance

THEOREM. For any noncomputable c.e. set A there are c.e. sets A_+ and A_- such that the following hold.

- A, A_+ and A_- are computably isomorphic.
- For $r \in \{ibT, sbT\}$, $A_- <_r A <_r A_+$.

So, in particular, ibT-equivalence and sbT-equivalence are not computably invariant.

Strongly bounded Turing reducibilities vs. truth-table-type reducibilities

$$
\begin{array}{ccc}
 & A \le_{bT} B & \\
\nearrow & & \nwarrow \\
A \le_{tt} B & & A \le_{sbT} B \\
\uparrow & & \uparrow \\
A \le_{btt} B & & A \le_{ibT} B \\
\uparrow & & \\
A \le_{m} B & & \\
\uparrow & & \\
A \le_{1} B & &
\end{array}
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\end{array}
$$

nonadaptive
computable bound

adaptive
identity bound

Strictness of $A \leq_{ibT} B \Rightarrow A \leq_{sbT} B \Rightarrow A \leq_{bT} B$

THEOREM (DHL 2001, BL 2006). Let A be a noncomputable c.e. set.

- For $A+1 = \{x+1 : x \in A\}$, $A+1 <_{ibT} A$ and $A+1 =_{sbT} A$.
- For $2A = \{2x : x \in A\}$, $2A <_{sbT} A$ and $2A =_{bT} A$.

PROOF.

$$\begin{aligned} A \leq_{ibT} A+1 \quad &\Rightarrow \quad A \text{ selfreducible} \\ &\Rightarrow \quad A \text{ computable} \end{aligned}$$

(And, similarly, for $A \leq_{sbT} 2A$.)

Incompatibility of truth-table and strongly bounded reducibilities

THEOREM. There are noncomputable c.e. sets A and B such that

- for $r \in \{1, m, btt, tt\}$, $A <_r B$
- for $r' \in \{ibT, sbT\}$, $B <_{r'} A$

PROOF (Finite Injury). Enumerate sets A and B such that

(i) $B \subseteq \{2x^2 : x \geq 0\} \cup \{2x^2 + 1 : x \geq 0\}$

(ii) $x \in A_{at\ s} \Leftrightarrow 2x^2 \in B_{at\ s}$

(iii) $2x^2 + 1 \in B_{at\ s} \Rightarrow x \in A_{at\ s}$

(iv) $\Re_e$: Φ_e total and nonadaptive $\Rightarrow \exists\, x\, (B(2x^2 + 1) \neq \Phi_e^A(2x^2 + 1))$

The partial orderings $(\mathrm{R}_{ibT}, \leq)$ and $(\mathrm{R}_{sbT}, \leq)$

For $r \in \{ibT, sbT\}$,

- $(\mathrm{R}_r, \leq)$ is a partial ordering with least element $\mathbf{0} = \{A : A \text{ computable}\}$
- $(\mathrm{R}_r, \leq)$ has no minimal nonzero elements
  (For noncomputable A, $\mathbf{0} < deg_{ibT}(A+1) < deg_{ibT}(A)$ and $\mathbf{0} < deg_{sbT}(2A) < deg_{sbT}(A)$.)

THEOREM (Barmpalias 2005). $(\mathrm{R}_{sbT}, \leq)$ does not possess any maximal elements (hence, in particular, there is no complete degree).

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THEOREM (Barmpalias 2005). $(\mathrm{R}_{sbT}, \leq)$ does not possess any maximal elements (hence, in particular, there is no complete degree).

The corresponding fact for $(\mathrm{R}_{ibT}, \leq)$ is trivial: For noncomputable A, $A <_{ibT} A - 1$ where $A - 1 := \{x - 1 : x \geq 1 \;\&\; x \in A\}$.
(Note that $(A-1)+1 =^* A$!)

Computable shifts

DEFINITION. A strictly increasing, computable function $f : \omega \longrightarrow \omega$ is called a computable shift. A shift f is nontrivial if $f(x) > x$ for some (hence for almost all) x, and f is unbounded if for any number k there is a number x such that $f(x) - x > k$.
For any set A and any shift f, the f-shift of A is defined by

$$A_f = \{f(x) : x \in A\}.$$

FACTS. (1) For any shift f, $x \leq f(x)$. Moreover, $f(x) - x$ is nondecreasing in x. So if f is unbounded then $\lim_{n\to\infty}(f(n) - n) = \sup_{n\to\infty}(f(n) - n) = \infty$.

(2) For any bounded shift f, $A_f =^* A + k$ for some $k \geq 0$ where $A + k = \{x + k : x \in A\}$.

(3) For any computable unbounded shift f there are computable unbounded shifts g and h such that $A_f = (A_g)_h$.

The Shift Lemma

SHIFT LEMMA (preliminary form). Let f be a computable shift and let A be a noncomputable c.e. set.

(i) $A \leq_1 A_f$ and $A_f =_m A$.

(ii) For nontrivial and bounded f, $A_f <_{ibT} A$ and $A_f =_{sbT} A$.

(iii) For unbounded f, $A_f <_{sbT} A$ and $A_f =_{bT} A$.

PROOF. As in case of the examples $A + 1$ and $2A$ of bounded and unbounded computable shifts, one shows that if (in (ii) or (iii)) A where reducible to A_f then A were selfreducible, hence computable.

The Shift Lemma

SHIFT LEMMA (final form). Let f be a computable shift and let A be a noncomputable c.e. set.

(i) $A \leq_1 A_f$ and $A_f =_m A$.

(ii) For nontrivial and bounded f, $A_f <_{ibT} A$ and $A_f =_{sbT} A$.
In fact, if B is a c.e. set such that $A \cap B = \emptyset$ and $A \leq_{ibT} A_f \dot\cup B$ then $A \leq_{ibT} B$.

(iii) For unbounded f, $A_f <_{sbT} A$ and $A_f =_{bT} A$.
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For a reducibility where the join of two degrees can be represented by the degree of the disjoint union of sets in these degrees, the above would show that deg(A) has the anti-cupping property via deg(A_f).

Barmpalias's Theorem

THEOREM. For any noncomputable c.e. set A there is a c.e. set $\hat{A}$ such that $A <_{sbT} \hat{A}$ (and $A <_{ibT} \hat{A}$). So $(\mathrm{R}_{sbT}, \leq)$ does not possess any maximal elements (hence does not have a greatest element).

PROOF (A-S, Ding, Fan, Merkle; Belanger). Fix an infinite computable subset B of A, and let $\hat{A}$ be the compressed version of $A \setminus B$ obtained by the order isomorphism $\pi : \omega \setminus B \to \omega$. Then $A \setminus B$ is a computable unbounded shift of $\hat{A}$ whence $A \setminus B <_{sbT} \hat{A}$ by the Shift Lemma. But, obviously, $A \setminus B =_{ibT} A$.

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Similarly, one obtains a set $\hat{A}$ as above such that A and $\hat{A}$ are computably isomorphic: Take two infinite computable disjoint subsets B and C of A. Let $\tilde{A}$ be the compressed version of $A \setminus (B \cup C)$ obtained by the order isomorphism $\pi : \omega \setminus (B \cup C) \to \omega \setminus C$, and let $\hat{A} = \tilde{A} \cup C$.

Another application of shifts: automorphisms

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PROOF. For the bounded shift $f(x) = x + 1$ mapping A to $A + 1$,

$$A \leq_{ibT} B \Leftrightarrow A + 1 \leq_{ibT} B + 1.$$

So f is ibT-degree invariant, and for the corresponding function $\mathbf{f}$ on the c.e. ibT-degrees

$$\mathbf{a} \leq \mathbf{b} \Leftrightarrow \mathbf{f}(\mathbf{a}) \leq \mathbf{f}(\mathbf{b}).$$

Moreover, $\mathbf{f}$ is onto since, for any set A, $A =^* (A - 1) + 1$ whence $deg_{ibT}(A) = \mathbf{f}(deg_{ibT}(A - 1))$.

Another application of shifts: automorphisms

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Moreover, $\mathbf{f}$ is onto since, for any set A, $A =^* (A - 1) + 1$ whence $deg_{ibT}(A) = \mathbf{f}(deg_{ibT}(A - 1))$.

Note that $\mathbf{f}$ generates the bounded (left and right) shifts. So the automorphism group of $(\mathrm{R}_{ibT}, \leq)$ is infinite.

Permitting vs. strongly bounded T-reducibility

PERMITTING LEMMA. If $B \leq_T A$ via permitting then $B \leq_{ibT} A$ (hence $B \leq_{sbT} A$).

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Conversely, any ibT-reduction can be represented by permitting.

REPRESENTATION LEMMA. Let A and B be noncomputable c.e. sets such that $B \leq_{(i+k)bT} A$. There are c.e. subsets $\hat{A}$ and $\hat{B}$ of A and B, resp., and computable 1-1 enumerations $\{a(n)\}_{n \geq 0}$ and $\{b(n)\}_{n \geq 0}$ of $\hat{A}$ and $\hat{B}$, resp., such that

- $\hat{A} =_{ibT} A$ and $\hat{B} =_{ibT} B$ and
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PROOF. Given an $(i+k)bT$-reduction $B = \Phi^A$, choose a computable increasing sequence $\{s_n\}_{n \geq 0}$ of expansionary stages such that between any two of these stages a number less than the previous length of agreement is enumerated into B and let $b(n)$ and $a(n)$ be the least numbers enumerated into B and A, resp., between stage s_n and s_{n+1}.

Splitting and strongly bounded T-reducibility

SPLITTING LEMMA (SL). For pairwise disjoint c.e. sets $A_0, \dots, A_n$ $(n \geq 1)$, $deg_r(A_0) \vee \cdots \vee deg_r(A_n) = deg_r(A_0 \cup \cdots \cup A_n)$ $(r \in \{ibT, sbT\})$.

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PROOF. $A_j \leq_{ibT} A_0 \cup \cdots \cup A_n$ by permitting. So, given B such that $A_j \leq_{(i+k_j)bT} B$, it suffices to show that $A_0 \cup \cdots \cup A_n \leq_{(i+k)bT} B$ where $k = max\ k_j$. But this is obviously true.

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DISTRIBUTIVITY LEMMA. Let $A_0, \dots, A_n$ $(n \geq 1)$ be pairwise disjoint c.e. sets, let $A = A_0 \cup \dots \cup A_n$, and let B be a c.e. set such that $B \leq_{(i+k)bT} A$. There is a c.e. set $\hat{B} =_{ibT} B$ and a splitting $\hat{B} = B_0 \cup \dots \cup B_n$ of $\hat{B}$ into pairwise disjoint c.e. sets B_j such that $B_j \leq_{(i+k)bT} A_j$ $(0 \leq j \leq n)$.

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PROOF. Choose $\hat{A}$, $\hat{B}$, $\{a(n)\}_{n \geq 0}$ and $\{b(n)\}_{n \geq 0}$ as in the Representation Lemma and let $B_j = \{b(n) : a(n) \in A_j\}$.

Bounded shifts and definability

THEOREM. Let $A \subseteq 2\omega$ ($A \subseteq 2\omega + 1$) be a noncomputable c.e. set and let B be a c.e. set such that $B \leq_{ibT} A$. T.f.a.e.

(i) B ibT-cups to A. I.e., there is a c.e. set $C <_{ibT} A$ such that

$$deg_{ibT}(A) = deg_{ibT}(B) \vee deg_{ibT}(C).$$

(ii) $B \not\leq_{ibT} A + 1$.

I.e., $deg_{ibT}(A + 1)$ is the greatest degree which does not cup to $deg_{ibT}(A)$.

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So, for A as above, the degrees $deg_{ibT}(A+k)$ $(k \geq 1)$ are definable with parameter $\deg_{ibT}(A)$.

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So, for A as above, the degrees $deg_{ibT}(A+k)$ $(k \geq 1)$ are definable with parameter $\deg_{ibT}(A)$.

COROLLARY. The theory of the c.e. ibT-degrees realizes infinitely many 2-types, hence is not ω-categorical.

Proof: (i) $\Rightarrow$ (ii)

It suffices to show that $A+1$ does not ibT-cup to A.

Assume

$$deg_{ibT}(A) = deg_{ibT}(A+1) \vee deg_{ibT}(C).$$

By RL w.l.o.g. we may assume that there are computable 1-1 enumerations $\{a(n)\}$ and $\{c(n)\}$ of A and C, resp., such that $a(n) \leq c(n)$. Split C into

$$\begin{array}{lcll} C_0 & = & \{c(n) : a(n) = c(n)\} & \cap A+1 = \emptyset \\ C_1 & = & \{c(n) : a(n) < c(n)\} & \leq_{ibT} A+1 \end{array}$$

Then, by SL,

$$A =_{ibT} A+1 \,\dot{\cup}\, C_0$$

whence $A \leq_{ibT} C_0 \leq_{ibT} C$ by the Shift Lemma (and SL).

Proof: (ii) $\Rightarrow$ (i)

Given $B \leq_{ibT} A$ such that $B \not\leq_{ibT} A+1$, it suffices to give $C <_{ibT} A$ such that

$$(*)\ deg_{ibT}(A) = deg_{ibT}(B) \vee deg_{ibT}(C).$$

By RL w.l.o.g. we may fix computable 1-1 enumerations $\{a(n)\}$ and $\{b(n)\}$ of A and B, resp., such that $a(n) \leq b(n)$. Split B into

$$\begin{array}{lcll} B_0 & = & \{b(n) : a(n) < b(n)\} & \leq_{ibT} A+1 \\ B_1 & = & \{b(n) : a(n) = b(n)\} & \not\leq_{ibT} A+1 \ \text{(by SL and by } B \not\leq_{ibT} A+1) \end{array}$$

Now, let $C = \{a(n) : a(n) < b(n)\}$. Since $A = B_1 \dot\cup C$, $(*)$ holds by SL. Finally, $C <_{ibT} A$ follows from the following observation.

PROPOSITION. Let D and E be disjoint c.e. sets such that $D \leq_{ibT} E$. Then $D \leq_{ibT} E+1$.

Bounded shifts and density

THEOREM (Barmpalias and Lewis 2006). There is a noncomputable c.e. set A such that

$$\forall\, B\ (A+1 \leq_{ibT} B \leq_{ibT} A \Rightarrow A+1 =_{ibT} B \text{ or } B =_{ibT} A).$$

Hence the partial ordering of the c.e. ibT-degrees is not dense.

The proof uses an infinite injury (tree) argument.

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THEOREM. There is a noncomputable c.e. set A such that the interval $[deg_{ibT}(A+1), deg_{ibT}(A)]$ is isomorphic to the countable atomless Boolean algebra.

The proof uses a finite injury argument.

Importing results from other degree structures

Some results about the weaker reducibilities $\leq_{bT}$ and $\leq_T$ carry over to $\leq_{sbT}$ and $\leq_{ibT}$ either

- directly or
- by using the fact that certain reductions there are ibT-reductions or
- by combining the results with some of the previously mentioned simple techniques and/or
- by using some additional facts on relations between the above reducibilities.

We give some examples.

Examples of direct imports

THEOREM. For $r \in \{ibT, sbT\}$, the partial ordering $(\mathrm{R}_r, \leq)$ is not total.

PROOF. There is a pair of T-incomparable c.e. sets, and any T-incomparable pair is r-incomparable.

THEOREM. For $r \in \{ibT, sbT\}$, there is a minimal pair of c.e. r-degrees.

PROOF. There is a T-minimal pair, and any such pair is r-minimal.

Examples of imports using the Splitting Lemma

THEOREM (DHL 2001 (?)). For $r \in \{ibT, sbT\}$, every nonzero c.e. r-degree splits.

PROOF. By Sacks's splitting theorem and by SL.

THEOREM (Barmpalias and Lewis). Every finite partial ordering is embeddable into $(\mathrm{R}_r, \leq)$ $(r \in \{ibT, sbT\})$.

PROOF. Given $n \geq 1$, it suffices to embed the partial ordering $\mathcal{P} = (\{\alpha : \alpha \subseteq \{0, \ldots, n\}\}, \subseteq)$. Let $A_0, \ldots, A_n$ be a T-independent sequence of c.e. sets, let $\hat{A}_0, \ldots, \hat{A}_n$ be pairwise disjoint sets such that $\hat{A}_i =_T A_i$ and let $B_\alpha = \cup_{i \in \alpha} \hat{A}_i$. Then, by T-independence and by SL, $B_\alpha \leq_r B_\beta \Leftrightarrow \alpha \subseteq \beta$.

Converting sbT into ibT

sbT-ibT-CONVERSION LEMMA. Let A and B be c.e. sets and $k \geq 1$. T.f.a.e.

1. $A \leq_{(i+k)bT} B$
2. $A + k \leq_{ibT} B$
3. $A \leq_{ibT} B - k$
4. $A + k' \leq_{ibT} B - k''$ $(k = k' + k'')$

(And $A + k =_{sbT} A$ and $B - k =_{sbT} B$.)

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sbT-ibT-CONVERSION LEMMA (General Form).

1. Let $A, B_0, \ldots, B_n$ be c.e. sets such that $A \leq_{sbT} B_0, \ldots, B_n$. There is a c.e. set $\hat{A} =_{sbT} A$ such that $\hat{A} \leq_{ibT} B_0, \ldots, B_n$.
2. Let $A_0, \ldots, A_n, B$ be c.e. sets such that $A_0, \ldots, A_n \leq_{sbT} B$. There is a c.e. set $\hat{B} =_{sbT} B$ such that $A_0, \ldots, A_n \leq_{ibT} \hat{B}$.

Converting sbT into ibT: meets and joins

The general form of the sbT-ibT-Conversion Lemma implies:

MEET LEMMA FOR ibT.

$$deg_{ibT}(A) = deg_{ibT}(B_0) \wedge \cdots \wedge deg_{ibT}(B_n)$$
$$\Downarrow$$
$$deg_{sbT}(A) = deg_{sbT}(B_0) \wedge \cdots \wedge deg_{sbT}(B_n)$$

JOIN LEMMA FOR ibT.

$$deg_{ibT}(B) = deg_{ibT}(A_0) \vee \cdots \vee deg_{ibT}(A_n)$$
$$\Downarrow$$
$$deg_{sbT}(B) = deg_{sbT}(A_0) \vee \cdots \vee deg_{sbT}(A_n)$$

Converting bT into sbT

bT-sbT-CONVERSION LEMMA. Let A and B be c.e. sets such that $A \leq_{bT} B$.

1. If the use of the reduction is bounded by the computable unbounded shift f then $A_f =_{bT} A$ and $A_f \leq_{ibT} B$
2. If R is an infinite computable subset of A and $\tilde{B}$ is the shifted version of B with domain R then, for $\hat{B} = (A \setminus R) \cup \tilde{B}$, $A \leq_{ibT} \hat{B}$ and $\hat{B} =_{sbT} B$

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bT-sbT-CONVERSION LEMMA (General Form).

1. Let $A, B_0, \ldots, B_n$ be c.e. sets such that $A \leq_{bT} B_0, \ldots, B_n$. There is a c.e. set $\hat{A} =_{bT} A$ such that $\hat{A} \leq_{ibT} B_0, \ldots, B_n$.
2. Let A, B be c.e. sets such that $A \leq_{sbT} B$. There is a c.e. set $\hat{B} =_{sbT} B$ such that $A \leq_{ibT} \hat{B}$.
   (In this case the more general corresponding version for sets $A_0, \ldots, A_n$ in place of A fails for $n \geq 1$.)

Converting bT into sbT: meets

The general form of the bT-sbT-Conversion Lemma implies:

MEET LEMMA FOR sbT.

$$deg_{sbT}(A) = deg_{sbT}(B_0) \wedge \cdots \wedge deg_{sbT}(B_n)$$
$$\Downarrow$$
$$deg_{bT}(A) = deg_{bT}(B_0) \wedge \cdots \wedge deg_{bT}(B_n)$$

A corresponding lemma for joins fails!

COROLLARY (Downey and Hirschfeldt). For $r \in \{ibT, sbT\}$, the partial ordering $(\mathrm{R}_r, \leq)$ is not a lower semi-lattice.

PROOF. Jockusch has shown that $(\mathrm{R}_{bT}, \leq)$ is not a lower semi-lattice and by the Meet Lemmas this carries over to $(\mathrm{R}_{sbT}, \leq)$ and $(\mathrm{R}_{ibT}, \leq)$.

Applications of the Meet Lemmas: Minimal Pairs

MINIMAL PAIR THEOREM. For c.e. sets A and B the following are equivalent.

- (A,B) is an ibT-minimal pair.
- (A,B) is an sbT-minimal pair.
- (A,B) is a bT-minimal pair.

Together with the conversion lemmas this implies:

NONBOUNDING THEOREM. For a c.e. set C the following are equivalent.

- C is ibT-nonbounding.
- C is sbT-nonbounding.
- C is bT-nonbounding.

Here a c.e. set C is r-nonbounding if there is no r-minimal pair (A,B) such that $A, B \leq_r C$.

1-types of the theories of the c.e. *ibT* and *sbT* degrees

The above results on minimal pairs and nonbounding degrees allow to transfer the proof of A-S and Soare that the theory of the c.e. *bT*-degrees has infinitely many 1-types to the c.e. *ibT*- and *sbT*-degrees.

THEOREM. The theories of $(\mathrm{R}_{ibT}, \leq)$ and $(\mathrm{R}_{sbT}, \leq)$ realize infinitely many 1-types, hence are not ω-categorical.

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PROOF (IDEA). By a $0'''$-argument A-S and Soare 1989 have shown that, for any $k \geq 1$, there are *bT*-nonbounding sets $A_0, \ldots, A_k$ which are pairwise *bT*-minimal pairs. By distributivity of the c.e. *bT*-degrees this implies that the $k+1$-atom Boolean algebra can be embedded into the interval $[\mathbf{0}, deg_{bT}(A_0) \vee \cdots \vee deg_{bT}(A_k)]$ whereas the $k+2$-atom Boolean algebra cannot be embedded into this interval (as Boolean algebras). If we replace the sets $A_0, \ldots, A_k$ by pairwise disjoint *bT*-equivalent c.e. sets then, by the Splitting and Distributivity Lemmas and by the Minimal Pair and Nonbounding Theorems, we can make the same observation for *ibT* and *sbT* in place of *bT*.

Pairs without joins

THEOREM (Barmpalias 2005; Fan and Lu 2005). There is a pair of c.e. sets A and B such that, for $r \in \{ibT, sbT\}$, $deg_r(A) \vee deg_r(B)$ does not exist.

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PROOF (Finite injury; $r = ibT$). Given V and ibT-reductions $A = \Phi^V$ and $B = \Psi^V$ we have to construct a c.e. set U such that $A, B \leq_{ibT} U$ and such that U meets the requirements

$$\Re_\Gamma : V \neq \Gamma^U$$

for all ibT-functionals Γ.

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ATTACK ON $\mathfrak{R}_\Gamma$:

- Fix x (fresh) and impose restraint on A, B, U.
- Wait for a stage s_1 such that $A \upharpoonright x+1 = \Phi^V \upharpoonright x+1$, $B \upharpoonright x+1 = \Psi^V \upharpoonright x+1$, and $V \upharpoonright x+1 = \Gamma^U \upharpoonright x+1$. Put x into A.
- Wait for Φ- and Ψ-recovery at a stage $s_2 > s_1$. $V \upharpoonright x+1$ must have changed. If $V \upharpoonright x$ has changed, complete the attack by putting x into U. Otherwise (i.e., if x has entered V) put x into U and B. (The latter forces V to change below x at a later stage.)

Maximal pairs

DEFINITION. A pair (A, B) of c.e. sets is an *r-maximal pair* if there is no c.e. set C such that $A, B \leq_r C$.

Note that, by the sbT-ibT-Conversion Lemma, ibT-maximal pairs and sbT-maximal pairs coincide.

THEOREM (Barmpalias 2005; Fan and Lu 2005). There is an ibT-maximal pair.

Construction of an ibT-maximal pair (A, B) (Y. Fan)

Given a computable enumeration $\{\hat{\Phi}_e\}_{e \geq 0}$ of the partial ibT-functionals, it suffices to meet the requirements

$$\mathfrak{R}_e : \; A \neq \hat{\Phi}_{e_1}^{W_{e_0}} \; \vee \; B \neq \hat{\Phi}_{e_2}^{W_{e_0}} \;\; (e = \langle e_0, e_1, e_2 \rangle)$$

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Split ω into infinitely many intervals I_e such that

$$|I_e| > |\bigcup_{e' < e} I_{e'}| \text{ (whence } 2 \cdot |I_e| > |\bigcup_{e' \leq e} I_{e'}|).$$

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Define A and B on I_e in such a way such that $\Re_e$ is met:

$$\exists\, x \in I_e \; (A(x) \neq \hat{\Phi}_{e_1}^{W_{e_0}}(x) \;\vee\; B(x) \neq \hat{\Phi}_{e_2}^{W_{e_0}}(x)).$$

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ACTION. If $\forall\, x \in I_e \; [A_s(x) = \hat{\Phi}_{e_1,s}^{W_{e_0,s}}(x) \;\&\; B_s(x) = \hat{\Phi}_{e_2,s}^{W_{e_0,s}}(x)]$ and if there is some $y \in I_e$ which has not yet put into A (or B) then put the least such y into A (or B).

More on maximal pairs

A-S, Ding, Fan and Merkle have shown:

- There is an ibT-maximal pair of m-complete sets.
- For any c.e. set C there is an ibT-maximal pair (A, B) such that $C \leq_{ibT} A, B$.
- There is a pair of c.e. sets (A, B) which is ibT-maximal and ibT-minimal.
- For a c.e. set C the following are equivalent.
  - $deg_T(C)$ is array noncomputable.
  - There is an ibT-maximal pair (A, B) such that $A =_T B =_T C$.
  - There is an ibT-maximal pair (A, B) such that $A =_T C$.
- If A is bT-complete then there is a c.e. set such that the pair (A, B) ist ibT-maximal. On the other hand, there is a T-complete set A which is not half of any ibT-maximal pair.

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- Is $(\mathrm{R}_{sbT}, \leq)$ rigid? Is there an automorphism of $(\mathrm{R}_{ibT}, \leq)$ which moves some degree (any nonzero degree) to an incomparable one?

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- Are the elementary theories of $(\mathrm{R}_{ibT}, \leq)$ and $(\mathrm{R}_{sbT}, \leq)$ undecidable? What are their degrees?
- Is every c.e. ibT (sbT) - degree branching?
- Is there any nondistributive lattice embeddable into $(\mathrm{R}_{ibT}, \leq)$ or $(\mathrm{R}_{sbT}, \leq)$?